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Constructing solutions for two-dimensional quasi-static problems of thermomechanics in terms of stresses for bodies with plane-parallel boundaries

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A methodology to construct solutions for two-dimensional quasi-static thermomechanical problems for bodies with plane-parallel boundaries (2D-QS thermomechanical problems) is proposed. This approach begins with selecting equations for the plane quasi-static thermoelasticity problem in terms of stresses. The methodology approximates the distribution of non-zero stress tensor components through the body's thickness using cubic polynomials, with coefficients expressed in terms of integral characteristics of the stress tensor components over the thickness variable and their specified boundary values on the body's lower and upper surfaces. Consequently, the original two-dimensional boundary problem is simplified to a one-dimensional boundary problem for the integral characteristics. For an infinite layer, solutions are found using the Fourier transform along the longitudinal coordinate, while for a strip plate, a finite integral transformation is applied along the transverse coordinate. General solutions for 2D-QS thermomechanical problems are formulated for the selected bodies under non-stationary volume forces and temperature fields. The resulting expressions for the stress tensor components are presented as convolutions of functions representing the boundary values on the bases (and end cross-sections for strip-plates) and functions describing homogeneous solutions to the one-dimensional boundary problems for the integral characteristics of the stress tensor components.

Keywords: *bodies with plane-parallel boundaries; layer; strip-plate; stress tensor components; cubic approximation; integral characteristics; Fourier transform; finite integral transform.*

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1. Introduction

Bodies with plane-parallel boundaries are commonly used as structural elements in various devices and modern engineering applications. Such bodies, exemplified by infinite layers (plates) and strip-plates, are subjected to the influence of volumetrically distributed non-stationary temperature fields and forces during the operation of respective devices. These two factors of thermal and force impact induce a corresponding thermoelastic state in the considered bodies. In the literature, particularly in [1–5], well-described methods for calculating the thermoelastic state in bodies of canonical shapes under the influence of temperature and force factors are based on one-dimensional models primarily utilizing displacement equations.

In the work [6], the original systems of equations for three- and two-dimensional dynamic thermoelasticity problems under stresses are formulated in Cartesian and cylindrical coordinate systems. The interrelated equations for the components of the dynamic stress tensor system are reduced to sequentially coupled wave equations for the corresponding combinations of these components. The key equations in these systems pertain to the first invariant of the stress tensor. In the case of quasi-static problems, such stress equations are significantly simplified. In the monograph [7], for instance, a two-dimensional coupled dynamic thermoelastic problem for an infinite layer is examined, and based on it, the thermo-mechanical behavior of such a layer under non-uniform electromagnetic action is investigated.

Considering the need to account for the influence of the nature of temperature and force factors on the operational modes, reliability, and longevity of plate-like structures, the development of effective methods for calculating 2D-QS stresses becomes a relevant applied and engineering task.

However, the calculation of quasi-static stresses in plate-like elements under the action of non-stationary temperature and force factors based on two-dimensional models is insufficiently explored and documented in the literature.

This article proposes an effective methodology for solving 2D-QS thermomechanical problems under stresses for bodies with plane-parallel boundaries in the presence of prescribed temperature fields and volume forces.

2. The system of original relationships for the 2D-QS thermoelasticity problem in stresses for bodies with plane-parallel boundaries

Consider a body with plane-parallel boundaries, referred to a Cartesian coordinate system $Ox_1x_2x_3$, where the plane x_1Ox_2 coincides with the mid-plane of the body. The body extends infinitely along the Ox_1 and Ox_2 axes, while along the Ox_3 axis, it has a constant thickness of $2h$. Here, h is half the thickness of the body. In the two-dimensional case, the body is subjected to the influence of a temperature field $T(x_{1,3}, t)$ and a volume force $\boldsymbol{F}(x_1, x_3, t) = \{F_1; 0; F_3\}$. These two thermal and force factors induce a thermoelastic state in the body. In the case of plane deformation, this state is described by the components $\sigma_{11}^{(s)}$, $\sigma_{22}^{(s)}$, $\sigma_{33}^{(s)}$, $\sigma_{13}^{(s)}$ of the quasi-static stress tensor.

For the 2D-QS thermoelasticity problem under stresses, the original system of equations for a body with plane-parallel boundaries takes the form

$$\Delta_1\psi^{(s)} = -\frac{\alpha E}{1-\nu}\Delta_1 T - \frac{1}{1-\nu}\left(\frac{\partial F_3}{\partial x_3} + \frac{\partial F_1}{\partial x_1}\right), \tag{1}$$

$$\Delta_1\sigma_{13}^{(s)} = -\frac{\partial^2\psi^{(s)}}{\partial x_1 \partial x_3} - \frac{\partial F_1}{\partial x_3} - \frac{\partial F_3}{\partial x_1}, \tag{2}$$

$$\frac{\partial \sigma_{11}^{(s)}}{\partial x_1} = \frac{\partial \sigma_{13}^{(s)}}{\partial x_3} - F_1, \tag{3}$$

$$\sigma_{33}^{(s)} = \psi^{(s)} - \sigma_{11}^{(s\_)}, \tag{4}$$

$$\sigma_{22}^{(s)} = \nu\psi^{(s)} - \alpha E T. \tag{5}$$

The boundary conditions on the surfaces $x_3 = \pm h$ of the body are

$$\frac{\partial \Psi^{(s)\pm}}{\partial x_1} + F_1^{\pm} + \frac{\partial \sigma_{13}^{(s)\pm}}{\partial z} = 0; \quad \sigma_{13}^{(s)\pm} = 0; \quad \sigma_{11}^{(s)\pm} = \Psi^{(s)\pm}, \tag{6}$$

and on the end surfaces $x_1 = \pm d$ of the body the boundary conditions are as follows:

$$\frac{\partial \Psi_*^{(s)\pm}}{\partial z} + F_{*3}^{\pm} + \frac{\partial \sigma_{*13}^{(s)\pm}}{\partial x_1} = 0; \quad \sigma_{*13}^{(s)\pm} = 0; \quad \sigma_{*33}^{(s)\pm} = \Psi_*^{(s)\pm}. \tag{7}$$

Let us construct solutions to the quasi-static thermoelasticity problems for an infinite layer with plane-parallel boundaries and a strip-plate. We will consider the system of equations (1)–(5) as the initial system, which, in the case of the layer, we will solve with the boundary conditions (6) on its bases $x_3 = \pm h$. Correspondingly, in the case of the strip-plate, additional conditions (7) need to be added to the boundary conditions (6) on its end intersections.

3. Definition of the thermoelastic state of an infinite layer

To solve the 2D-QS thermomechanical problem described by the system of equations (1)–(5) for an infinite layer along the x_1 coordinate, we approximate the distribution of functions $\psi^{(s)}(x_1, x_3)$ and $\sigma_{13}^{(s)}(x_1, x_3)$ over the thickness coordinate x_3 with cubic polynomials

$$\psi^{(s)} = \sum_{j=1}^{4} \psi_{j-1}^{(s)}(x_1)\, x_3^{j-1}, \tag{8}$$

$$\sigma_{13}^{(s)}=\sum_{j=1}^{4}\alpha_{13(j-1)}^{(s)}(x_1)\,x_3^{j-1}. \tag{9}$$

Then the coefficients $\psi_{j-1}^{(s)}$, $\alpha_{13(j-1)}^{(s)}$ of the approximation polynomials (8), (9) are expressed in terms of integral characteristics

$$N^{(s)}=\frac{1}{2h}\int_{-h}^{h}\psi^{(s)}dx_3,\quad M^{(s)}=\frac{3}{2h^2}\int_{-h}^{h}\psi^{(s)}x_3\,dx_3,$$
$$N_{13}^{(s)}=\frac{1}{2h}\int_{-h}^{h}\sigma_{13}^{(s)}dx_3,\quad M_{13}^{(s)}=\frac{3}{2h^2}\int_{-h}^{h}\sigma_{13}^{(s)}x_3\,dx_3 \tag{10}$$

of functions $\psi^{(s)}(x_1,x_3)$ and $\sigma_{13}^{(s)}(x_1,x_3)$ along the thickness coordinate x_3, and the boundary values of these functions are given as follows

$$\psi_{(0)}^{(s)}=\frac{3}{2}N^{(s)}-\frac{1}{4}\psi_{*}^{(s)},\quad \psi_{(1)}^{(s)}=\frac{5}{2h}M^{(s)}-\frac{3}{4h}\psi_{**}^{(s)},$$
$$\psi_{(2)}^{(s)}=\frac{3}{4h^2}\psi_{*}^{(s)}-\frac{3}{2h^2}N^{(s)},\quad \psi_{(3)}^{(s)}=\frac{5}{4h^3}\psi_{**}^{(s)}-\frac{5}{2h^2}M^{(s)},$$
$$\alpha_{13(0)}^{(s)}=\frac{3}{2}N_{13}^{(s)},\quad \alpha_{13(1)}^{(s)}=\frac{5}{2h}M_{13}^{(s)},$$
$$\alpha_{13(2)}^{(s)}=-\frac{3}{2h^2}N_{13}^{(s)},\quad \alpha_{13(3)}^{(s)}=-\frac{5}{2h^3}M_{13}^{(s)}. \tag{11}$$

Here

$$\psi_{*}^{(s)}=\psi^{(s)+}+\psi^{(s)-},\quad \psi_{**}^{(s)}=\psi^{(s)+}-\psi^{(s)-}.$$

The system of equations for determining the integral characteristics $N^{(s)}$, $M^{(s)}$, $N_{13}^{(s)}$, $M_{13}^{(s)}$ (analogues of forces and moments) is obtained by averaging the first two equations (1), (2) of the system (1)–(5) over the thickness coordinate x_3 and these equations multiplied by x_3, according to the formulas (10). In the case of a body with plane-parallel boundaries, the integral characteristics $N^{(s)}$, $M^{(s)}$, $N_{13}^{(s)}$, $M_{13}^{(s)}$ are determined from a system of one-dimensional equations,

$$\left(\frac{\partial^2}{\partial x_1^2}-\frac{3}{h^2}\right)N^{(s)}=\Phi_1^{(s)}-\frac{3}{2h^2}\psi_{*}^{(s)}, \tag{12}$$

$$\left(\frac{\partial^2}{\partial x_1^2}-\frac{15}{h^2}\right)M^{(s)}=\Phi_2^{(s)}-\frac{15}{2h^2}\psi_{**}^{(s)}, \tag{13}$$

$$\left(\frac{\partial^2}{\partial x_1^2}-\frac{3}{h^2}\right)N_{13}^{(s)}=\frac{1}{2h}\left[\frac{\partial\psi_{**}^{(s)}}{\partial x_1}+(F_1^{+}-F_1^{-})\right]+\frac{\partial F_3^{(1)}}{\partial x_1}, \tag{14}$$

$$\left(\frac{\partial^2}{\partial x_1^2}-\frac{15}{h^2}\right)M_{13}^{(s)}=\frac{3}{2h}\left[\frac{\partial\psi_{*}^{(s)}}{\partial x_1}+(F_1^{+}+F_1^{-})\right]-\frac{3}{h}\left(\frac{\partial N^{(s)}}{\partial x_1}+F_1^{(1)}\right)+\frac{\partial F_3^{(2)}}{\partial x_1}. \tag{15}$$

Here

$$\Phi_1^{(s)}=-\frac{\alpha E}{1-\nu}\left\{\frac{\partial^2T_1}{\partial x_1^2}+\frac{1}{2h}\left[\left(\frac{\partial T}{\partial x_3}\right)^{+}-\left(\frac{\partial T}{\partial x_3}\right)^{-}\right]\right\}-\frac{1}{1-\nu}\left[\frac{1}{2h}(F_3^{+}-F_3^{-})+\frac{\partial F_1^{(1)}}{\partial x_1}\right],$$

$$\Phi_2^{(s)}=-\frac{\alpha E}{1-\nu}\left\{\frac{\partial^2T_2}{\partial x_1^2}+\frac{3}{2h}\left[\left(\frac{\partial T}{\partial x_3}\right)^{+}+\left(\frac{\partial T}{\partial x_3}\right)^{-}\right]-\frac{3}{2h^2}(T^{+}-T^{-})\right\}$$
$$-\frac{1}{1-\nu}\left[\frac{3}{2h}(F_3^{+}+F_3^{-})-F_3^{(1)}+\frac{\partial F_1^{(2)}}{\partial x_1}\right],$$

where

$$T_n=\frac{2n-1}{2h^n}\int_{-h}^{h}T\,x_3^{n-1}dx_3,\quad F_i^{(n)}=\frac{2n-1}{2h^n}\int_{-h}^{h}F_i\,x_3^{n-1}dx_3,\quad (n=1,2,\quad i=1,3)$$

are integral characteristics of temperature and components of the volumetric force vector

$$\boldsymbol{F} = \{F_1; 0; F_3\}, \quad \left(\frac{\partial T}{\partial x_3}\right)^{\pm} = \frac{\partial T(x_1, \pm h, t)}{\partial x_3}, \quad F_i^{\pm} = F_1(x_1, \pm h, t).$$

The boundary values of $\psi^{(s)\pm}$ of the function $\Psi^{(s)}$ on the surfaces $x_3 = \pm h$, which are included in the system of equations (12)–(15), are found from the system of equations

$$\frac{\partial \psi_*^{(s)}}{\partial x_1} + (F_1^+ + F_1^-) - \frac{10}{h} M_{13}^{(s)} = 0, \tag{16}$$

$$\frac{\partial \psi_{**}^{(s)}}{\partial x_1} + (F_1^+ - F_1^-) - \frac{6}{h} N_{13}^{(s)} = 0, \tag{17}$$

obtained by averaging the boundary conditions (6) in accordance with the relations (10).

To solve the system of equations (12)–(15) for the integral characteristics of the functions $\psi^{(s)}(x_1, x_3)$ and $\sigma_{13}^{(s)}(x_1, x_3)$, we apply the direct Fourier transform $\tilde{f}(\xi) = \frac{1}{2\pi}\int_{-\infty}^{\infty} f(x_1)\, e^{i\xi x_1}\, dx_1$. Here, ξ is the Fourier transform parameter. Then the Fourier transforms of the functions $\psi^{(s)}(x_1, x_3)$ and $\sigma_{13}^{(s)}(x_1, x_3)$ are determined by the expressions:

$$\tilde{\psi}^{(s)}(\xi, x_3) = \sum_{j=1}^{4} \tilde{\psi}_{(j-1)}^{(s)}(\xi)\, x_3^{j-1}, \tag{18}$$

$$\tilde{\sigma}_{13}^{(s)}(\xi, x_3) = \sum_{j=1}^{4} \tilde{\alpha}_{13\,(j-1)}^{(s)}(\xi)\, x_3^{j-1}. \tag{19}$$

The Fourier transforms of the coefficients $\tilde{\psi}_{(j-1)}^{(s)}(\xi)$ and $\tilde{\alpha}_{13\,(j-1)}^{(s)}(\xi)$ of the approximation polynomials (18)–(19) are determined by the relationships:

$$\tilde{\psi}_0^{(s)} = \frac{3}{2}\tilde{N}^{(s)} - \frac{1}{4}\tilde{\psi}_*^{(s)}, \quad \tilde{\psi}_1^{(s)} = \frac{5}{2h}\tilde{M}^{(s)} - \frac{3}{4h}\tilde{\psi}_{**}^{(s)},$$
$$\tilde{\psi}_2^{(s)} = \frac{3}{4h^2}\tilde{\psi}_*^{(s)} - \frac{3}{2h^2}\tilde{N}^{(s)}, \quad \tilde{\psi}_3^{(s)} = \frac{5}{4h^3}\tilde{\psi}_{**}^{(s)} - \frac{5}{2h^2}\tilde{M}^{(s)},$$
$$\tilde{\alpha}_{13\,(0)}^{(s)} = \frac{3}{2}\tilde{N}_{13}^{(s)}, \quad \tilde{\alpha}_{13\,(1)}^{(s)} = \frac{5}{2h}\tilde{M}_{13}^{(s)}, \quad \tilde{\alpha}_{13\,(2)}^{(s)} = -\frac{3}{2h^2}\tilde{N}_{13}^{(s)},$$
$$\tilde{\alpha}_{13(3)}^{(s)} = -\frac{5}{2h^3}\tilde{M}_{13}^{(s)}, \quad \tilde{\psi}_*^{(s)} = \tilde{\psi}^{(s)+} + \tilde{\psi}^{(s)-}, \quad \tilde{\psi}_{**}^{(s)} = \tilde{\psi}^{(s)+} - \tilde{\psi}^{(s)-}. \tag{20}$$

The Fourier transforms of the coefficients $\psi_*^{(s)}$, $\psi_{**}^{(s)}$, $N^{(s)}$, $M^{(s)}$, $N_{13}^{(s)}$, $M_{13}^{(s)}$ are given by the following formulas, accordingly

$$\tilde{\psi}_+^{(s)}(\xi) = -\frac{10\big(3\tilde{F}_1^{(1)} - i\xi h \tilde{F}_3^{(2)}\big)}{i\xi(\xi^2h^2 + 30)} - (\tilde{F}_1^+ + \tilde{F}_1^-) - \frac{30}{(\xi^2 + \frac{3}{h^2})(\xi^2h^2 + 30)}\tilde{\Phi}_1^{(s)}, \tag{21}$$

$$\tilde{\psi}_{**}^{(s)}(\xi) = -\frac{h^2\xi^2 + 6}{(h^2+3)i\xi}(\tilde{F}_1^+ - \tilde{F}_1^-) - \frac{6h}{h^2+3}\tilde{F}_3^{(1)}, \tag{22}$$

$$\tilde{N}^{(s)}(\xi) = \frac{\frac{15}{hi\xi}\big(\frac{3}{h}\tilde{F}_1^{(1)} - i\xi\tilde{F}_3^{(2)}\big) - \frac{3}{2}(\tilde{F}_1^+ + \tilde{F}_1^-) - (60 + \xi^2h^2)\tilde{\Phi}_1^{(s)}}{(\xi^2 + \frac{3}{h^2})(\xi^2 + \frac{30}{h^2})h^2}, \tag{23}$$

$$\tilde{M}^{(s)}(\xi) = -\frac{(h^2+3)\tilde{\Phi}_2^{(s)} - \frac{15}{2h^2}\left[\frac{h^2\xi^2+6}{i\xi}(\tilde{F}_1^+ - \tilde{F}_1^-) + 6h\tilde{F}_3^{(1)}\right]}{(h^2+3)(\xi^2 + \frac{15}{h^2})}, \tag{24}$$

$$\tilde{N}_{13}^{(s)}(\xi) = \frac{\frac{h^2(\xi^2-1)+3}{2h}(\tilde{F}_1^+ - \tilde{F}_1^-) - i\xi h^2\tilde{F}_3^{(1)}}{(h^2+3)(\xi^2 + \frac{3}{h^2})}, \tag{25}$$

$$\tilde{M}_{13}^{(s)}(\xi) = \frac{1}{\xi^2 + \frac{15}{h^2}}\left\{(\tilde{F}_1^+ + \tilde{F}_1^-)\left[\frac{3(i\xi - 1)}{2h} - \frac{9i\xi}{2h^3(\xi^2 + \frac{3}{h^2})}\right]\right.$$

$$+\left(\frac{3}{h}\tilde{F}_1^{(1)}-i\xi\tilde{F}_3^{(2)}\right)\left[1-\frac{15}{\xi^2h^2+30}+\frac{45}{h^4}\frac{1}{(\xi^2+\frac{3}{h^2})(\xi^2+\frac{30}{h^2})}\right]$$
$$\left.-\frac{3}{h}\frac{i\xi(45+\xi^2h^2)}{(\xi^2+\frac{3}{h^2})(\xi^2+30)}\tilde{\Phi}_1^{(s)}\right\}. \quad (26)$$

Here

$$\tilde{\Phi}_1^{(s)}(\xi)=-\frac{\alpha E}{1-\gamma}\left\{-\xi^2\tilde{T}_1+\frac{1}{2h}\left[\left(\frac{d\tilde{T}}{dx_3}\right)^+-\left(\frac{d\tilde{T}}{dx_3}\right)^-\right]\right\}+\frac{1}{\gamma-1}\left[\frac{1}{2h}(\tilde{F}_3^+-\tilde{F}_3^-)+i\xi\tilde{F}_1^{(1)}\right],$$
$$\tilde{\Phi}_2^{(s)}(\xi)=-\frac{\alpha E}{1-\gamma}\left\{-\xi^2\tilde{T}_2+\frac{3}{2h}\left[\left(\frac{d\tilde{T}}{dx_3}\right)^++\left(\frac{d\tilde{T}}{dx_3}\right)^-\right]-\frac{3}{2h^2}(\tilde{T}^+-\tilde{T}^-)\right\}$$
$$-\frac{1}{1-\gamma}\left[\frac{3}{2h}(\tilde{F}_3^++\tilde{F}_3^-)-\tilde{F}_3^{(1)}+i\xi\tilde{F}_1^{(2)}\right];$$
$$\tilde{T}_n(\xi)=\frac{2n-1}{2h^n}\int_{-h}^{h}T(\xi,x_3)\,x_3^{n-1}dx_3,\quad \tilde{F}_i^{(n)}=\frac{2n-1}{2h^n}\int_{-h}^{h}F_i(\xi,x_3)\,x_3^{n-1}dx_3.$$

With the found transforms $\tilde{\psi}_*^{(s)}$, $\tilde{\psi}_{**}^{(s)}$, $\tilde{N}^{(s)}$, $\tilde{M}^{(s)}$, $\tilde{N}_{13}^{(s)}$, $\tilde{M}_{13}^{(s)}$, according to the expressions (21)–(26), the transforms of the stress tensor components $\tilde{\psi}^{(s)}=\tilde{\sigma}_{11}^{(s)}+\tilde{\sigma}_{33}^{(s)}$ and $\tilde{\sigma}_{13}^{(s)}$ are determined using the relationships (18)–(19). With the known transform $\tilde{\sigma}_{13}^{(s)}$, the transforms of $\tilde{\sigma}_{11}^{(s)}$ components of the stress tensor are determined by the formula:

$$\tilde{\sigma}_{11}^{(s)}(\xi,x_3,t)=-\frac{1}{i\xi}\left(\frac{d\tilde{\sigma}_{13}}{dx_3}+\tilde{F}_1\right). \quad (27)$$

After that, we find the transformants $\tilde{\sigma}_{22}^{(s)}$ and $\tilde{\sigma}_{33}^{(s)}$ of the components $\sigma_{22}^{(s)}$ and $\sigma_{33}^{(s)}$ of the stress tensor $\hat{\sigma}$ from the expressions

$$\tilde{\sigma}_{22}^{(s)}=\gamma\tilde{\psi}^{(s)}, \quad (28)$$
$$\tilde{\sigma}_{33}^{(s)}=\tilde{\psi}^{(s)}-\tilde{\sigma}_{11}^{(s)}. \quad (29)$$

Applying the inverse Fourier transform $f(x_1)=\frac{1}{2\pi}\int_{-\infty}^{\infty}\tilde{f}(\xi)\,e^{-i\xi x_1}d\xi$, we express the solution to the quasistatic problem of thermomechanics in stresses for an infinite layer with plane-parallel boundaries using the expressions (28)–(29).

4. Determining the thermo-stressed state of a strip-plate

Let us construct an approximate solution to the quasi-static problem of thermomechanics to determine the thermo-stressed state of a strip-plate. We will start with the system of equations (1)–(5), representing the 2D-QS problem of thermoplasticity for bodies with flat parallel boundaries. In the absence of external force loads on the end sections $x_1=\pm d$, by averaging the boundary conditions (7) on these sections and taking into account the conjugation conditions of functions

$$\Psi(d,\tau)=\Psi(1,\tau),\quad \Psi(d,\tau)=\Psi(-1,\tau),$$
$$\Psi(-d,\tau)=\Psi(1,\tau),\quad \Psi(-d,\tau)=\Psi(-1,\tau).$$
$$\sigma_{13}(d,\tau)=\sigma_{13}(1,\tau),\quad \sigma_{13}(d,\tau)=\sigma_{13}(-1,\tau),$$
$$\sigma_{13}(-d,\tau)=\sigma_{13}(1,\tau),\quad \sigma_{13}(-d,\tau)=\sigma_{13}(-1,\tau) \quad (30)$$

and values of functions $\Psi(x_1,\tau)$ and $\sigma_{13}(x_1,\tau)$ (stress tensor components) at the corners of the cross-sectional rectangle of the strip, we obtain the following boundary conditions with respect to the coordinate x_1 on the functions $\Psi^{\pm}$, N, M, N_{13}, M_{13}:

$$\psi^{\pm}(\pm d,t)=0,\quad N_{13}(\pm d,t)=0,\quad M_{13}(\pm d,t)=0,$$
$$N(\pm d,t)=F_3^+(\pm d,t)-F_3^-(\pm d,t),$$
$$M(\pm d,t)=\frac{h^2}{5}\left\{\frac{1}{2}\left[F_3^+(\pm d,t)-F_3^-(\pm d,t)\right]+F_3^{(1)}(\pm d,t)\right\}+\frac{h}{10}\left(\frac{dN_{13}}{dx_1}\right)_{x_1=\pm d}. \quad (31)$$

In accordance with the inhomogeneous boundary conditions (31) on the functions N and M, we represent the solutions of the first two equations of the system (12)–(13) in the form:

$$N(x_1,t)=\frac{1}{2}\left\{[N_*(d,t)+N_{**}(-d,t)]+\frac{x_1}{d}\left[N_*(d,t)-N_{**}(-d,t)\right]\right\}+N_0(x_1,t), \tag{32}$$

$$M(x_1,t)=\frac{1}{2}\left\{[M_*(d,t)+M_{**}(-d,t)]+\frac{x_1}{d}\left[M_*(d,t)-M_{**}(-d,t)\right]\right\}+M_0(x_1,t). \tag{33}$$

Then the functions $N_0(x_1,t)$ and $M_0(x_1,t)$ can be defined from equations

$$\left(\frac{d^2}{dx_1^2}-\frac{3}{h^2}\right)N_0=\Phi_1-\frac{3}{2h^2}(\psi^++\psi^-)+\frac{3}{2h^2}\left[(N_*+N_{**})+\frac{x_1}{d}(N_*-N_{**})\right], \tag{34}$$

$$\left(\frac{d^2}{dx_1^2}-\frac{15}{h^2}\right)M_0=\Phi_2-\frac{15}{2h^2}(\psi^+-\psi^-)+\frac{15}{2h^2}\left[(M_*+M_{**})+\frac{x_1}{d}(M_*+M_{**})\right] \tag{35}$$

under homogeneous boundary conditions on the surfaces $x_1=\pm d$.

Here

$$N_*=N(d,t),\quad N_{**}=N(-d,t),\quad M_*=M(d,t),\quad M_{**}=M(-d,t).$$

To solve the system of interrelated equations (31), (34)–(35), we will apply a finite integral transformation along the coordinate x_1.

In this process, when constructing the kernels of the finite integral transformation and their eigenvalues, we will take into account that the differential operators with respect to the coordinate x_1 in the original equations (31), (34)–(35) for the functions N_0, M_0, M_{13}, $\psi^\pm$ are the same, and the boundary conditions are of the same type. Therefore, considering the mutual orthogonality of eigenfunctions, from the equations (34)–(35), taking into account the boundary conditions (31), we will find the kernel of the finite integral transformation along the coordinate x_1

$$K(\alpha_k,x_1)=\frac{\sin\alpha_k(x_1+d)}{\sqrt{d}},\quad \text{where}\quad \alpha_k=\frac{\pi k}{2d}. \tag{36}$$

Based on the found expressions for N, M, N_{13}, M_{13}, and $\psi^\pm$ functions, ψ and σ_{13} can be expressed as follows:

$$\begin{aligned}\psi(x_1,x_3,t)=&\frac{3}{2}\left(1-\frac{x_3^2}{h^2}\right)N(x_1,t)+\frac{5}{2}\left(\frac{x_3}{h}-\frac{x_3^3}{h^3}\right)M(x_1,t)\\&-\frac{1}{4}\left(1-3\frac{x_3^2}{h^2}\right)\psi_*(x_1,t)-\frac{1}{4}\left(3\frac{x_3}{h}-5\frac{x_3^3}{h^3}\right)\psi_{**}(x_1,t),\end{aligned} \tag{37}$$

$$\sigma_{13}(x_1,x_3,t)=\frac{3}{2}\left(1-\frac{x_3^2}{h^2}\right)N_{13}(x_1,t)+\frac{5}{2}\left(\frac{x_3}{h}-\frac{x_3^3}{h^3}\right)M_{13}(x_1,t). \tag{38}$$

Here $\Psi_*(x_1,t)=\Psi^++\Psi^-$, $\Psi_{**}(x_1,t)=\Psi^+-\Psi^-$.

To determine the component σ_{11} of the stress tensor, it is necessary to solve the equation (3) with known expressions (37) and (38) for ψ and σ_{13}. Applying a finite integral transform along the x_1 coordinate, taking into account zero boundary conditions, and considering the mutual orthogonality of eigenfunctions, we find the expression for the component σ_{11}

$$\begin{aligned}\tilde{\sigma}_{11}(x_1,p)=\sum_{k=1}^{\infty}\frac{c_3^2}{p^2+c_3^2\alpha_k^2}&\left[\rho\frac{\nu(1+\nu)}{E}p^2\tilde{\tilde{\psi}}\,(\alpha_k,p)-\rho\,\alpha(1+\nu)p^2\tilde{\tilde{T}}(\alpha_k,p)\right.\\&\left.+\int_{-d}^{d}\frac{d\tilde{F}_1(x_1,p)}{dx_1}K(\alpha_k,x_1)dx_1\right]K(\alpha_k,x_1).\end{aligned} \tag{39}$$

The functions $N_{13}^{(s)}$, $M_{13}^{(s)}$, found using the finite integral transform along the x_1 coordinate, will be equal to

$$N_{13}^{(s)}(x_1)=-\sum_{k=1}^{\infty}\frac{\int_{-d}^{d}\frac{dF_3^{(1)}(x_1)}{dx_1}K(\alpha_k,x_1)\,dx_1}{\alpha_k^2+\frac{6}{h^2}}K(\alpha_k,x_1), \tag{40}$$

$$M_{13}^{(s)}(x_1)=\sum_{k=1}^{\infty}\left[\frac{3}{h}\tilde{F}_1^{(1)}(\alpha_k)-\int_{-d}^{d}\frac{dF_3^{(2)}(x_1)}{dx_1}K(\alpha_k,x_1)dx_1+\frac{3}{2h^{\frac{3}{2}}d}\frac{1-(-1)^k}{\alpha_k}(N_*-N_{**})\right]\frac{K(\alpha_k,x_1)}{\alpha_k^2+\frac{30}{h^2}}, \tag{41}$$

and functions $N^{(s)}$ and $M^{(s)}$ can be written as

$$N^{(s)}=\frac{1}{2}\left[(N_*^{(s)}+N_{**}^{(s)})+\frac{x_1}{d}(N_*^{(s)}-N_{**}^{(s)})\right] \\ +\sum_{k=1}^{\infty}\left\{\frac{3}{2h^2}(\tilde{\psi}^{+(s)}+\tilde{\psi}^{-(s)})-\tilde{\Phi}_1^{(s)}-\frac{3}{h^2\sqrt{d}\alpha_k}\left[N_*^{(s)}-(-1)^kN_{**}^{(s)}\right]\right\}\frac{K(\alpha_k,x_1)}{(\alpha_k^2+\frac{3}{h^2})}, \tag{42}$$

$$M^{(s)}=\frac{1}{2}\left[(M_*^{(s)}+M_{**}^{(s)})+\frac{x_1}{d}(M_*^{(s)}-M_{**}^{(s)})\right] \\ +\sum_{k=1}^{\infty}\left\{\frac{15}{2h^2}(\tilde{\psi}^{+(s)}-\tilde{\psi}^{-(s)})-\tilde{\Phi}_2^{(s)}-\frac{15}{h^2\sqrt{d}\alpha_k}\left[M_*^{(s)}-(-1)^kM_{**}^{(s)}\right]\right\}\frac{K(\alpha_k,x_1)}{(\alpha_k^2+\frac{3}{h^2})}. \tag{43}$$

Here

$$N_*^{(s)}=\frac{h}{6}(F_{3*}^{+}-F_{3*}^{-});\qquad N_{**}^{(s)}=\frac{h}{6}(F_{3**}^{+}-F_{3**}^{-}),$$
$$M_*^{(s)}=\frac{h^2}{5}\left[\frac{1}{2}(F_{3*}^{+}-F_{3*}^{-})+F_{3*}^{(1)}\right]+\frac{h}{10}\left(\frac{dN_{13}}{dx_1}\right)_{x_1=d},$$
$$M_{**}^{(s)}=\frac{h^2}{5}\left[\frac{1}{2}(F_{3**}^{+}-F_{3**}^{-})+F_{3**}^{(1)}\right]+\frac{h}{10}\left(\frac{dN_{13}}{dx_1}\right)_{x_1=-d}.$$

With known functions $N_{13}^{(s)}$ and $M_{13}^{(s)}$, the functions $\psi^{(s)\pm}$ are found from expressions (6) and (7) through direct integration with respect to x_1, taking into account the zero boundary values of the function $\psi^{\pm}$ at the end faces $x_1=\pm d$. The functions $\psi^{(s)}$ and $\sigma_{1s}^{(s)}$ are determined by the relations (37) and (38), and the stress components $\sigma_{11}^{(s)}$, $\sigma_{22}^{(s)}$, $\sigma_{33}^{(s)}$ are determined by the relations (3)–(5).

5. Conclusions

The 2D-QS problems of thermomechanics in stresses for bodies with flat-parallel boundaries have been formulated. To construct approximate solutions for the formulated quasi-static thermomechanical problems for an infinite layer and a strip-plate, approximations of stress tensor components' distributions along the thickness variable with cubic polynomials were utilized. As a result, the original two-dimensional boundary problems for the layer and strip-plate regarding stress tensor components were reduced to one-dimensional boundary problems for their integral characteristics. The solutions to these problems for an infinite layer were found using Fourier transform along the longitudinal coordinate. For the strip-plate, a finite integral transform along the transverse coordinate was applied. Expressions for stress tensor components were obtained as convolutions of functions describing the boundary values of these components on the layer bases and the end cross-sections of the strip-plate, as well as homogeneous solutions to one-dimensional boundary problems for integral characteristics.

The developed methodology for determining stress tensor components in bodies with flat-parallel boundaries significantly simplifies the analytical general solutions of the formulated 2D-QS thermomechanical problems and their numerical analysis for specific temperature distributions and volumetric forces.

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Побудова розв'язків двовимірних квазістатичних задач термомеханіки у напруженнях для тіл з плоскопаралельними границями

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Запропоновано методику побудови розв'язків двовимірних квазістатичних задач термомеханіки для тіл з плоскопаралельними границями. За вихідну вибрана система рівнянь плоскої квазістатичної задачі термопружності у напруженнях. Методика ґрунтується на апроксимації розподілу відмінних від нуля компонент тензора напружень по товщинній змінній тіла кубічними поліномами. Коефіцієнти апроксимаційних поліномів виражаються через інтегральні по товщинній змінній характеристики компонент тензора напружень та їх задані граничні значення на нижній і верхній основах тіла. У результаті вихідна двовимірна крайова задача на компоненти тензора напружень зведена до одновимірної крайової задачі на їх інтегральні характеристики. Для безмежного шару записано розв'язки одновимірної крайової задачі з використанням інтегрального перетворення Фур'є за поздовжньою координатою. У випадку смуги-пластини для знаходження розв'язку використано скінчене інтегральне перетворення за поперечною координатою. Записано загальні розв'язки двовимірних крайових задач термомеханіки для розглядуваних тіл за наявності в них нестаціонарних об'ємних сил та температурного поля. Вирази шуканих компонент тензора напружень подано у вигляді згорток функцій, що описують задані граничні значення цих компонент на основах розглядуваних тіл, а також на торцевих перетинах у випадку смуги-пластини і функцій, що описують однорідні розв'язки одновимірних крайових задач на інтегральні характеристики компонент тензора напружень.

Ключові слова: *тіла з плоскопаралельними границями; шар; смуга-пластина; компоненти тензора напружень; кубічна апроксимація; інтегральні характеристики; перетворення Фур"є; скінчене інтегральне перетворення.*