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Solution to the Fokker–Plank equation in the path integral method

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A Fokker–Plank equation of multiple variables corresponding to a system of SDE is considered. Solution for transition probability density is written in a form of path integral. It is shown that the proposed path integral brings a known result received by a different approach for Heston model. Differences of results based on path integral given in a number of papers were also pointed out.

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1. Introduction

Stochastic processes are fundamental in research of various phenomena: physical, financial and others [1–3]. In order to model stochastic processes one uses stochastic differential equations (SDE), that are also known as Langevin equations [1,2,4–6]. SDE contain stochastic variables that are usually based on stochastic Wiener process (Brownian motion). System of SDE is compared to Fokker–Plank equation (FP) that allows to determine the transition probability density of stochastic process variables (the Cauchy problem for FP equation) and other stochastic characteristics. As it is known, application of SDE results in stochastic integrals (differentials of functions of stochastic variables) for which a special rules of stochastic calculus were developed, like Itô, Stratonovich, and some intermediate schemes [4,5]. As a result, each schema of SDE system corresponds to a slightly different Fokker–Plank equation.

The path integral method [7–10] plays an important role in obtaining solutions of FP equations. Actually, integral over Wieners measure is a path integral for the simplest stochastic equation, that describes Brownian motion. Path integral methods are well developed for one-dimensional SDE, where path integrals are built based on solutions to FP equations as well as by variable substitution in Wieners measure [9–11].

In the case of stochastic processes of multiple variables, there are a lot of inconsistencies in building solution to FP equation using path integral method. In many papers building a solution to Fokker–Plank equation for transitional probability density is done by analogy like for the propagator of the Schrödinger equation in quantum mechanics for imaginary time [7]. In [12] for Fokker–Plank equation a known solution was used for propagator of Schrödinger equation [13] in the multi-dimensional curved space. The FP equation was received based on SDE in a generalized schema that contains Itô and Stratonovich exceptional cases. Let us note, that solution [13] for Schrödinger equation in form of Feynman integral in multi-dimensional curved space differs from a similar solution [14] by a coefficient near a term with scalar curviness in Lagrange function. Other generalizations of Feynman integral known for propagator in curved space [15] differ by a mentioned term with scalar curviness as well as contain different structure terms.

In of [16] the operator of Fokker–Plank equation is considered as operator of some quantum-mechanical system with a "classic action" expression received by means of Weyl rule of ordering of "coordinate" and "momentum" operators. Noted "classic action" was used to build solution to FP equation in the form of Feynman path integral. A similar approach was used with application of other ordering rules of "coordinate" and "momentum" operators in [17]. Let us note, that in such approach solution to FP equation is written using various path integrals. The Siggia–Rose–Janssen–deDominicis (MSRJD) method is also used to determine solution to FP equation in the form of path integral [15].

In our work a solution to multi-dimensional FP equation in the form of path integral was built based on methods described in [11, 18, 19]. We consider FP equation corresponding to Itô's stochastic calculus that is generally used in problems of financial engineering. Based on received path integral we found a solution to the two-dimensional Heston model which is used in research of assets and derivatives pricing in financial engineering. Obtained solutions of Heston model are compared to solutions found in [20] using different approaches. We also point out differences in results based on path integrals built in a number of works mentioned above.

2. Construction of path integral for the Fokker–Plank equation

Let us consider a general stochastic model of multiple variables which is given by system of SDE, which are also known as Langevin equations [1, 2, 4]:

$$dX_i(\tau) = A_i(X(\tau))\,d\tau + \sum_j^n B_{ij}(X(\tau))\,dW_j(\tau), \quad i \in \{1,\dots,n\}. \tag{1}$$

Here we introduce notation of stochastic variables $X_i(\tau)$ and variables of Wieners processes $W_i(\tau)$, $i \in \{1,\dots,n\}$, $\tau \in [t_0, t]$. Values $A_i(X(\tau))$, $i \in \{1,\dots,n\}$ denote components of drift vector which depend on set of variables $X_i(\tau)$, $B_{ij}(X(\tau))$ is matrix of dimension $n \times n$, elements of which set the volatility of stochastic variables ($(i,j) \in \{1,\dots,n\}$). Wiener processes $W_i(\tau)$, $i \in \{1,\dots,n\}$ are considered to be uncorrelated and fulfill the following known equations:

$$dW_i(\tau)\,dW_j(\tau) = \delta_{ij}\,d\tau, \quad (i,j) \in \{1,\dots,n\}.$$

For the SDE system (1) using Itô's stochastic calculus we receive the following FP equation for transitional probability density of stochastic process [1, 4]

$$\frac{\partial K(x,t)}{\partial t} = \frac{1}{2}\sum_{ij}^n \frac{\partial^2 \Sigma_{ij}(x)K(x,t)}{\partial x_i \partial x_j} - \sum_i^n \frac{\partial A_i(x)K(x,t)}{\partial x_i}. \tag{2}$$

Here $K(x,t)$, $A_i(x)$, $\Sigma_{ij}(x)$ depends on set of variables x_i, $(i,j) \in \{1,\dots,n\}$. Diffusion matrix $\mathbf{\Sigma}(x)$ is defined by matrix $\mathbf{B}(x)$:

$$\Sigma_{ij}(x) = \sum_k^n B_{ik}B_{jk}(x), \quad (i,j) \in \{1,\dots,n\}. \tag{3}$$

In the system of Stratonovich stochastic calculus or for a number of intermediate systems the FP equation differs by drift quantities $A_i(x)$, $i \in \{1,\dots,n\}$ [4, 12, 15, 16] hence the following series covers all mentioned use cases.

Let us write FP equation (2) in operator form

$$\frac{\partial K(x,t)}{\partial t} = -\mathcal{H}K(x,t),$$

where $\mathcal{H}$ is the operator of FP equation

$$\mathcal{H} = -\frac{1}{2}\sum_{ij}^n \frac{\partial^2}{\partial x_i \partial x_j}\Sigma_{ij}(x) + \sum_i^n \frac{\partial}{\partial x_i}A_i(x). \tag{4}$$

For transitional probability density let us build solution in a form of exponential operator

$$K(x, x_0, t, t_0) = e^{-(t-t_0)\mathcal{H}} \prod_i^n \delta(x_i - x_{0i}). \tag{5}$$

Formula (5) gives a solution to Cauchy problem for FP equation. By generalizing one-dimensional case [11], let us write operator (4) in the equal form

$$\mathcal{H} = -\frac{1}{2}\sum_k^n \hat{\mathcal{P}}_k^2 + U(x). \tag{6}$$

The following operators are denoted:

$$\hat{\mathcal{P}}_k = \sum_i^n B_{ik}(x)\frac{\partial}{\partial x_i} - p_k(x), \quad k \in \{1,\ldots,n\}. \tag{7}$$

Values $U(x)$ and $p_k(x)$, $k \in \{1,\ldots,n\}$ we shall determine from equality of operators (4) and (6). After a number of transformations we receive a system of equations for determining $p_k(x)$, $k \in \{1,\ldots,n\}$:

$$\sum_k^n B_{ik}(x)\bigg(p_k(x) + \sum_j^n \frac{\partial B_{jk}(x)}{\partial x_j}\bigg) = A_i(x) - \frac{1}{2}\sum_{k,j}^n \frac{\partial B_{ik}(x)}{\partial x_j} B_{jk}(x), \quad i \in \{1,\ldots,n\}, \tag{8}$$

and also expression for $U(x)$:

$$\begin{gathered} U(x) = \frac{1}{2}\sum_k^n \tilde{p}_k(x)^2 + \frac{1}{2}\sum_{k,i}^n \frac{\partial \tilde{p}_k(x)}{\partial x_i} B_{ik}(x), \\ \tilde{p}_k(x) = p_k(x) + \sum_i^n \frac{\partial B_{ik}(x)}{\partial x_i}, \quad k \in \{1,\ldots,n\}. \end{gathered} \tag{9}$$

From the system of equations (8) it results that representation (6) is valid if matrix $B_{ik}(x)$, $(i,k) \in \{1,\ldots,n\}$ is invertable. Then the system of equations (7) unequivocally is solvable for p_k, $k \in \{1,\ldots,n\}$. For exponential operator (5) taking into account form (6) we shall apply Gaussian path integral [11, 18, 19]

$$e^{-(t-t_0)\mathcal{H}} = \int \mathcal{D}q(\tau)\exp\bigg(-\frac{1}{2}\sum_k^n \int_{t_0}^t q_k^2(\tau)\,d\tau\bigg)\mathrm{T}\exp\bigg(-\sum_k^n \int_{t_0}^t q_k(\tau)\,\hat{\mathcal{P}}_k\,d\tau - \int_{t_0}^t U(x)\,d\tau\bigg). \tag{10}$$

The following is denoted:

$$\mathcal{D}q(\tau) = \prod_k^n \mathcal{D}q_k(\tau), \quad \mathcal{D}q_k(\tau) = \prod_{\tau=t_0}^t \sqrt{\frac{d\tau}{2\pi}}\,dq_k(\tau), \tag{11}$$

symbol 'T' denotes chronological ordering of operators.

Let us consider differential operator in exponent (10) in details

$$\sum_i^n \bigg(\sum_k^n B_{ik}(x)\,q_k(\tau)\bigg)\frac{\partial}{\partial x_i}. \tag{12}$$

Since terms $\sim \frac{\partial}{\partial x_i}$ for various $i \in \{1,\ldots,n\}$ in (12) do not commute in pairs, it is impossible to "untangle" them. For this let us introduce new set of variables y_i, $i \in \{1,\ldots,n\}$ based on the following system of equations:

$$x_i = \varphi_i(y_1, y_2, \ldots, y_n), \quad i \in \{1,\ldots,n\}. \tag{13}$$

Here $\varphi_i(y_1, y_2, \ldots, y_n)$, $i \in \{1,\ldots,n\}$ is a set of some linearly independent functions. Actions of differential operators on arbitrary function $\Phi(x_1, x_2, \ldots, x_n)$ of variables $x_1, x_2, \ldots, x_n$ we shall give as follows:

$$\sum_i^n B_{ik}(x)\,\frac{\partial}{\partial x_i}\Phi(x_1, x_2, \ldots, x_n) = \frac{\partial}{\partial y_k}\Phi\big(\varphi_1, \varphi_2, \ldots, \varphi_n\big), \quad k \in \{1,\ldots,n\}. \tag{14}$$

Here functions φ_i, $i \in \{1,\ldots,n\}$ (14) we omit variables for simplicity. Based on (13) and (14) we receive the following solution:

$$\frac{\partial \varphi_i}{\partial y_k} = B_{ik}(\varphi), \quad (i,k) \in \{1,\ldots,n\}, \tag{15}$$

where in matrix $B_{ik}(\varphi)$ elements we performed substitution $x_i = \varphi_i$, $i \in \{1,\ldots,n\}$ according to (13). Since determinant $\|\frac{\partial \varphi_i}{\partial y_k}\| \neq 0$, the variables y_k, $k \in \{1,\ldots,n\}$ are linearly independent and mapping (13) is bijective.

As a result we receive the following form of operator action in exponent (10)

$$\mathrm{T}\exp\bigg(-\sum_{k=1}^n \int_{t_0}^t q_k(\tau)\,d\tau\frac{\partial}{\partial y_k} - \int_{t_0}^t F(\varphi(y))\,d\tau\bigg)\prod_{i=1}^n \delta(\varphi_i(y) - x_{0i}), \tag{16}$$

where $F(x)$ is a some function of n variables.

Differential operators easily "untangle" and for (16) we receive

$$\exp\left(-\int_{t_0}^{t} F(\phi(y(\tau)))\,d\tau\right)\prod_{i=1}^{n}\delta(\varphi_i(y(t_0))-x_{0i}). \tag{17}$$

Here $y(\tau)$ denotes a set of variables $y_k(\tau)$, $k\in\{1,\dots,n\}$:

$$y_k(\tau)=y_k-\int_{\tau}^{t} q_k(\tau_1)\,d\tau_1,\quad k\in\{1,\dots,n\}.$$

In path integral (10), after transformations (12)–(17), we perform the following variable substitution:

$$\varphi_i\biggl(\dots,y_k-\int_{\tau}^{t} q_k(\tau_1)\,d\tau_1,\dots\biggr)=x_i-\int_{\tau}^{t}\nu_i(\tau_1)\,d\tau_1,\quad i\in\{1,\dots,n\}. \tag{18}$$

Differentiating (18) for τ we receive:

$$\sum_k B_{ik}(x(\tau))\,q_k(\tau)=\nu_i(\tau),\quad i\in\{1,\dots,n\}. \tag{19}$$

Here variables $x_i(\tau)$, $i\in\{1,\dots,n\}$ are given by right sides of equalities (18).

As a result for transitional probability density (5) we receive the following solution in the form of path integral

$$K(x,x_0,t,t_0)=\int\mathcal{D}\nu(\tau)\,J\exp\left(-\frac{1}{2}\sum_{i,k}^{n}\int_{t_0}^{t}\nu_i(\tau)\,\Sigma_{ij}^{-1}(x(\tau))\,\nu_k(\tau)\,d\tau\right)$$
$$\times\exp\left(\sum_{i,k}^{n}\int_{t_0}^{t}p_k(x(\tau))\,B_{ki}^{-1}(x(\tau))\,\nu_i(\tau)\,d\tau-\int_{t_0}^{t}U(x(\tau))\,d\tau\right)\prod_{i=1}^{n}\delta\biggl(x_i-x_{0i}-\int_{t_0}^{t}\nu_i(\tau)\,d\tau\biggr). \tag{20}$$

Here values $p_k(x)$, $k\in\{1,\dots,n\}$, $U(x)$ are defined in formulas (8) and (9). Functional Jacobian J of variable substitution (18), (19) is given in Appendix A. Element of functional measure $\mathcal{D}\nu(\tau)$ is defined in formula (11).

Let us perform a number of transformation in formula (20). In particular, the term in exponent (20) and the term from Jacobian (49), taking into account equation for $p_k(x(\tau))$ (8), we shall transform like the following

$$\sum_{i,k}^{n}p_k(x(\tau))\,B_{ki}^{-1}(x(\tau))\,\nu_i(\tau)+\frac{1}{2}\sum_{kij}^{n}\frac{\partial B_{kj}(x(\tau))}{\partial x_k}B_{ji}^{-1}(x(\tau))\,\nu_i(\tau)$$
$$=\sum_{ij}^{n}\nu_i\,\Sigma_{ij}^{-1}(x(\tau))\biggl(A_j(x(\tau))-\frac{1}{2}\sum_{k}^{n}\frac{\partial\Sigma_{kj}(x(\tau))}{\partial x_k}\biggr). \tag{21}$$

Matrix $\Sigma(x)$ is defined in formula (3). For $U(x)$ (9) after analogous transformations we receive

$$U(x(\tau))=\frac{1}{2}\sum_{ij}^{n}A_i^c(x(\tau))\,\Sigma_{ij}^{-1}(x(\tau))\,A_j^c(x(\tau))+\frac{1}{2}\sum_{i}^{n}\frac{\partial A_i^c(x(\tau))}{\partial x_i}$$
$$+\frac{1}{8}\sum_{ijk}^{n}\frac{\partial B_{jk}(x(\tau))}{\partial x_j}\frac{\partial B_{ik}(x(\tau))}{\partial x_i}+\frac{1}{4}\sum_{ijk}^{n}B_{ik}(x(\tau))\frac{\partial^2 B_{jk}(x(\tau))}{\partial x_i\partial x_j}. \tag{22}$$

Here the following is denoted

$$A_i^c(x(\tau))=A_i(x(\tau))-\frac{1}{2}\sum_{j}^{n}\frac{\partial\Sigma_{ij}(x(\tau))}{\partial x_j}.$$

The last three terms in (22) can also be reduced to the form

$$\frac{1}{2}\sum_{i}^{n}\frac{\partial A_i(x(\tau))}{\partial x_i}-\frac{1}{8}\sum_{ij}^{n}\frac{\partial^2 W_{ij}(x(\tau))}{\partial x_i\,\partial x_j}-\frac{1}{8}\sum_{ijk}^{n}\frac{\partial B_{jk}(x(\tau))}{\partial x_i}\frac{\partial B_{ik}(x(\tau))}{\partial x_j}.$$

Term (21) and the first term in (22) we shall unite with the term $\sim\nu_i(\tau)\,\nu_j(\tau)$ in exponent (20) and as the result we write the following:

$$K(x,x_0,t,t_0)=\int\tilde{\mathcal{D}}\nu(\tau)\exp\left(-\int_{t_0}^{t}L(x(\tau))\,d\tau\right)\prod_{i=1}^{n}\delta\left(x_i-x_{0i}-\int_{t_0}^{t}\nu_i(\tau)\,d\tau\right), \tag{23}$$

$$L(x(\tau))=L_0(x(\tau))+U_0(x(\tau)),$$

$$L_0(x(\tau))=\frac{1}{2}\sum_{i,j}^{n}\big(\nu_i(\tau)-A_i^c(x(\tau))\big)\,\Sigma_{ij}^{-1}(x(\tau))\,\big(\nu_j(\tau)-A_j^c(x(\tau))\big)\,d\tau,$$

$$U_0(x(\tau)))=\frac{1}{2}\sum_{i}^{n}\frac{\partial A_i(x(\tau))}{\partial x_i}-\frac{1}{8}\sum_{ij}^{n}\frac{\partial^2 W_{ij}(x(\tau))}{\partial x_i\,\partial x_j}-\frac{1}{8}\sum_{ijk}^{n}\frac{\partial B_{jk}(x(\tau))}{\partial x_i}\frac{\partial B_{ik}(x(\tau))}{\partial x_j}.$$

Here $U_0(x(\tau))$ has a missing first term ($\sim A_i^c\,A_j^c$) if compared with formula (22), and also element of functional measure in (23) is equal to:

$$\tilde{\mathcal{D}}\nu(\tau)=\prod_{\tau=t_0}^{t}\frac{1}{\det(B(x(\tau)))}\prod_{k}^{n}\mathcal{D}\nu_k(\tau),\quad \mathcal{D}\nu_k(\tau)=\prod_{\tau=t_0}^{t}\sqrt{\frac{d\tau}{2\pi}}\,d\nu_k(\tau). \tag{24}$$

This way the path integral (23) give the solution for transitional probability density of FP equation (2), which corresponds to the system of SDE (1). Path integral (23) is given in the "velocity" space $\nu_i(\tau)$ [7,11,18,19], where the solutions with "coordinates" $x_i(\tau)$, $i\in\{1,\dots,n\}$ is defined in (18). One can be convinced that for one-dimensional case of for a system of independent one-dimensional SDE the path integral (23) coincides with the one given in [11].

Let us make a few comments regarding building of path integral for FP equation based on quantum-mechanical analogy [7,16,17]. In particular, based on (23) one can obtain description of "quantum-mechanical system" by transition to imaginary time $t\to \mathrm{i}t$ and by substitution $\nu_k(\tau)\to\nu_k(\tau)/\mathrm{i}$, $k\in\{1,\dots,n\}$ (i is imaginary unit). Then inside exponent (23) we receive

$$-\int_{t_0}^{t}L(x(\tau))\,d\tau\to \mathrm{i}\int_{t_0}^{t}L_{QM}(x(\tau))\,d\tau,$$

where

$$L_{QM}(x)=\frac{1}{2}\sum_{i,j}^{n}\big(\nu_i-\mathrm{i}A_i^c(x)\big)\Sigma_{ij}^{-1}(x)\big(\nu_j-\mathrm{i}A_j^c(x)\big)-U_0(x) \tag{25}$$

is a Langrange function of some "classic system". Based on (25) we find Hamilton function by ususal means [7]

$$H_{QM}(x)=\sum_{i}^{n}\nu_i\,\frac{\partial L_{QM}(x)}{\partial\nu_i}-L_{QM}(x),$$

and "momentum":

$$p_i=\frac{\partial L_{QM}(x)}{\partial\nu_i},\quad i\in\{1,\dots,n\}.$$

As a result we find Hamilton function of the system

$$H_{QM}(x)=\frac{1}{2}\sum_{i,j}^{n}p_i\,\Sigma_{ij}(x)\,p_j+\mathrm{i}\sum_{i}^{n}A_i^c(x)\,p_i+U_0(x). \tag{26}$$

Hamilton function (26) compares to FP operator (4)

$$\mathcal{H}=\frac{1}{2}\sum_{i,j}^{n}\hat{p}_i\,\hat{p}_j\,\Sigma_{kj}(x)+\mathrm{i}\sum_{i}^{n}\hat{p}_i\,A_i(x), \tag{27}$$

where following "momentum" operators are denoted:

$$\hat{p}_i=-\mathrm{i}\,\frac{\partial}{\partial x_i},\quad i\in\{1,\dots,n\}. \tag{28}$$

Application of Feynman integral in phase space [7,16,17] for Hamilton function (26) and transforming back to real time allows to obtain solution to FP equation which is equivalent to (23).

Approach of works [7,16,17] for building path integral consists in establishing a form of Hamilton function (26) and application of Feynman integral in phase space. Procedure of "quantification" (replacement of "momentum" with corresponding operators (28)) in combination with the certain rule of "ordering" must lead to operator of FP equation (27). However application of such an approach in this case does not give expected results. It can be seen from application of various ordering rules of operators [16, 17] for function (26) lead to scalar expressions built using matrix elements $\Sigma_{ij}(x)$ and their derivatives. At the same time $U_0(x)$ in (26) contains expressions with elements of matrix $B_{ij}(x)$ and their derivatives, which are not possible to express in terms of $\Sigma_{ij}(x)$. This way operator (27) cannot be received from (26) using any of the ordering operators of "momentum" and "coordinates".

3. Heston model

Let us consider Heston model given by the following two SDE:

$$\begin{aligned} dS(\tau) &= \mu S(\tau)\, d\tau + S(\tau)\sqrt{V(\tau)}\, dW_s(\tau), \\ dV(\tau) &= \kappa(\theta - V(\tau))\, d\tau + \sigma\sqrt{V(\tau)}\, dW_v(\tau). \end{aligned} \tag{29}$$

First equation models price dynamics $S(\tau)$, where volatility contains stochastic value $V(\tau)$, dynamic of which is given by the second equation. Equation of price dynamics is a generalization of geometric Brownian motion and the second equation is known as CIR stochastic process [21]. Wiener process in equations (29) are considered correlated:

$$\langle dW_s(\tau)\rangle = \langle dW_v(\tau)\rangle = 0, \quad \langle dW_s(\tau)^2\rangle = \langle dW_v(\tau)^2\rangle = d\tau,$$
$$\langle dW_s(\tau)\, dW_v(\tau)\rangle = \rho\, d\tau, \quad -1 \leqslant \rho \leqslant 1.$$

Heston model is an extension of Black–Scholes option pricing model and takes into account stochastic volatility. Heston obtained solution for option price in mentioned model [21] by means of selection of characteristic function and giving form structure of option price. Equation for option price in Heston model was investigated by various methods [21] like: characteristic functions, transformations of Laplace and Fourier. Let us note that option price equation $C(t)$ is an inverted Kolmogorov equation for discount average of the payment function

$$C(t) = e^{-r(t-t_0)} \int_K^\infty \int_0^\infty K(S, S_0, V, V_0, t, t_0)(S - K)\, dV\, dS,$$

where r is interest rate, K is strike price [21].

FP equation for transitional probability density of stochastic variable in (29) model where investigated much less. In particular FP equation for Heston model was solved in work [20], where Fourier and Laplace transforms of S and V variables respectively where applied.

In equations (1) Wieners processes are given correlated, for that matter let us perform transformation to uncorrelated Wieners processes $W_1(\tau)$, $W_2(\tau)$ in (29):

$$W_s(\tau) = \sqrt{1-\rho^2} W_1(\tau) + \rho W_2(\tau), \quad W_v(\tau) = W_2(\tau), \quad \langle W_1(\tau)\, W_2(\tau)\rangle = 0.$$

System of equation of Heston model (29) we leave in the form

$$\begin{pmatrix} dS(\tau) \\ dV(\tau) \end{pmatrix} = \begin{pmatrix} \mu\, S(\tau) \\ \kappa(\theta - V(\tau)) \end{pmatrix} d\tau + \sqrt{V(\tau)} \begin{pmatrix} S(\tau)\sqrt{1-\rho^2} & S(\tau)\,\rho \\ 0 & \sigma \end{pmatrix} \begin{pmatrix} dW_1(\tau) \\ dW_2(\tau) \end{pmatrix}. \tag{30}$$

Based on (30) we write drift vector and diffusion matrix of the model:

$$\mathbf{A}(\tau) = \begin{pmatrix} \mu\, S(\tau) \\ \kappa(\theta - V(\tau)) \end{pmatrix}, \quad \mathbf{B}(\tau) = \sqrt{V(\tau)} \begin{pmatrix} S(\tau)\sqrt{1-\rho^2} & S(\tau)\,\rho \\ 0 & \sigma \end{pmatrix}. \tag{31}$$

3.1. Case of uncorrelated Wieners processes $\rho = 0$

Let us consider the case of uncorrelated Wieners processes and set $\rho = 0$ in (31). Substituting values (31) in general formula (23) we receive:

$$L(\tau) = L_0(\tau) + U_0(V), \tag{32}$$

$$L_0(\tau) = \frac{1}{2}\frac{\big(\nu_1(\tau) - \mu S(\tau) + S(\tau)V(\tau)\big)^2}{S(\tau)^2 V(\tau)} + \frac{1}{2}\frac{\big(\nu_2(\tau) + \frac{\sigma^2}{2} - \kappa(\theta - V(\tau))\big)^2}{\sigma^2 V(\tau)}, \tag{33}$$

$$U_0(V) = -\frac{\kappa}{2} + \frac{\mu}{2} - \frac{\sigma^2}{32 V(\tau)} - \frac{3V(\tau)}{8}.$$

Element of functional measure (24) is the following:

$$\tilde{\mathcal{D}}\nu(\tau) = \Bigg(\prod_{\tau=t_0}^{t} \frac{1}{\sigma\, S(\tau)\, V(\tau)}\Bigg) \prod_{k=1}^{2} \mathcal{D}\nu_k(\tau), \quad \mathcal{D}\nu_k(\tau) = \prod_{\tau=t_0}^{t} \sqrt{\frac{d\tau}{2\pi}}\, d\nu_k(\tau), \quad k \in \{1,2\}. \tag{34}$$

Dependency on $S(\tau)$ is only in the first term $L_0(\tau)$ (33), so let us split expression $L_0(\tau)$ intot two parts

$$L_0(\tau) = L_{0S}(\tau) + L_{0V}(\tau),$$

where

$$L_{0S}(\tau) = \frac{1}{2}\frac{\big(\nu_1(\tau) - \mu S(\tau)\big)^2}{S(\tau)^2 V(\tau)} + \frac{\nu_1(\tau)}{S(\tau)}, \tag{35}$$

$$L_{0V}(\tau) = \frac{1}{2}\frac{\nu_2(\tau)^2}{\sigma^2 V(\tau)} + \left(\frac{\kappa}{\sigma^2} + \frac{1-\alpha}{2V(\tau)}\right)\nu_2(\tau) + \frac{1}{8}\frac{(1-\alpha)^2\sigma^2}{V(\tau)} + \frac{1}{2}V(\tau) - \mu + \frac{1}{2}(1-\alpha)\kappa + \frac{\kappa^2}{2\sigma^2}. \tag{36}$$

Here denoted $\alpha = \frac{2\theta\kappa}{\sigma^2}$.

Path integrals we calculate successively, first over variable $S(\tau)$. In particular, for the second term in (35) we receive

$$\exp\left(-\int_{t_0}^{t} \frac{\nu_1(\tau)}{S(\tau)}\, d\tau\right) = \exp\left(-\int_{t_0}^{t} \frac{\dot{S}(\tau)}{S(\tau)}\, d\tau\right) = \frac{S_0}{S}.$$

Taking into account the form of the first term in (35) we perform variable substitution in path integral $\nu_1(\tau) \to q_1(\tau)$ using formula

$$\frac{\nu_1(\tau) - \mu S(\tau)}{S(\tau)\sqrt{V(\tau)}} = q_1(\tau). \tag{37}$$

Let us consider the first order differential equation (37) relative to function $S(\tau)$ ($\dot{S}(\tau) = \nu_1(\tau)$) with initial condition $S(t) = S$, where functions $V(\tau), q_1(\tau)$ are considered unknown. Solution of equation (37) is the following

$$S(\tau) = \exp\left(-\mu(t-\tau) - \int_{\tau}^{t} \sqrt{V(\tau_1)}\, q_1(\tau)\, d\tau_1\right) S.$$

For corresponding δ-function in formula (23) we receive

$$\delta\left(\exp\left(-\mu(t-t_0) - \int_{t_0}^{t} \sqrt{V(\tau)}\, q_1(\tau)\, d\tau\right) S - S_0\right). \tag{38}$$

By calculating Jacobian of variable substitution (37) using approach given in Appendix A we receive

$$J_S = \left\| \frac{\delta\nu_1(\tau)}{\delta q_1(\tau')} \right\| = \left(\prod_{\tau=t_0}^{t} S(\tau)\sqrt{V(\tau)}\right)\sqrt{\frac{S_0}{S}}. \tag{39}$$

In order to calculate path integral for $q_1(\tau)$, $\tau \in [t_0, t]$ we use Fourier transform for δ-function (38), which is given in Appendix B. As a result for path integral of variable $q_1(\tau)$, $\tau \in [t_0, t]$ we receive the following

$$\begin{aligned} I_S(V) &= \sqrt{\left(\frac{S_0}{S}\right)^3} \int \mathcal{D}q_1(\tau)\, e^{-\frac{1}{2}\int_{t_0}^{t} q_1(\tau)^2 d\tau} \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{1}{S_0}\, e^{-\mathrm{i}k\ln(S_0/S)} e^{-\mathrm{i}\,k\left(\mu(t-t_0) + \int_{t_0}^{t} q_1(\tau)\sqrt{V(\tau)}\, d\tau\right)} dk \\ &= \sqrt{\frac{S_0}{S^3}} \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-\mathrm{i}\,k(\ln(S_0/S) + \mu(t-t_0))}\, e^{-\frac{k^2}{2}\int_{t_0}^{t} V(\tau) d\tau} dk. \end{aligned}$$

Multiplier from Jacobian (39) is partially reduces with an analogous in (34). So transitional probability density is determined using following path integral

$$K(S, S_0, V, V_0, t, t_0) = \int \tilde{\mathcal{D}}\nu_2(\tau)\, e^{-\int_{t_0}^{t}(L_{0V}(\tau) + U_0(V))\, d\tau} I_S(V)\, \delta\left(V - V_0 - \int_{t_0}^{t} \nu_2(\tau)\, d\tau\right).$$

Here element of functional measure is given by the following expression

$$\tilde{\mathcal{D}}\nu_2(\tau) = \prod_{\tau=t_0}^{t} \sqrt{\frac{d\tau}{2\pi\sigma^2 V(\tau)}}\, d\nu_2(\tau).$$

For the second term in $L_{0V}(\tau)$ (36) we receive ($\dot{V}(\tau) = \nu_2(\tau)$)

$$\exp\left(- \int_{t_0}^{t} \left(\frac{\kappa}{\sigma^2} + \frac{1-\alpha}{2V(\tau)} \right) \nu_2(\tau)\, d\tau \right) = \exp\left(- \frac{(V - V_0)\kappa}{\sigma^2} \right) \left(\frac{V}{V_0} \right)^{-\frac{1}{2}(1-\alpha)}.$$

In the next step, using structure of the first term (36) we shall perform the following variable substitution

$$\frac{\nu_2(\tau)}{\sigma\sqrt{V(\tau)}} = q_2(\tau). \tag{40}$$

We shall solve differential equation for $V(\tau)$ with initial condition $V(t) = V$:

$$\frac{\dot{V}(\tau)}{\sigma\sqrt{V(\tau)}} = \dot{z}_2(\tau), \quad z_2(\tau) = z_2 - \int_{\tau}^{t} q_2(\tau_1)\, d\tau_1,$$

receiving

$$V(\tau) = \frac{1}{4}\sigma^2 z_2(\tau)^2.$$

Jacobian of variable substitution (40) we calculate using approach given in Appendix A. Thus we receive the following:

$$J_V = \left\| \frac{\delta \nu_2(\tau)}{\delta q_2(\tau')} \right\| = \left(\prod_{\tau=t_0}^{t} \sigma\sqrt{V(\tau)} \right) \sqrt[4]{\frac{V_0}{V}}.$$

Taking into account mentioned transformations for transitional probability density we receive

$$K(S, S_0, V, V_0, t, t_0) = \frac{4}{\sigma^2} e^{\frac{1}{2}(\mu+\alpha\kappa)(t-t_0)} \sqrt{\frac{S_0}{S^3}} e^{-\frac{(V-V_0)\kappa}{\sigma^2}} \left(\frac{V}{V_0} \right)^{\frac{1}{4}(2\alpha-3)} \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-\mathrm{i}\, k(\ln(S_0/S)+\mu(t-t_0))}$$
$$\times \int \mathcal{D}q_2(\tau)\, e^{-\frac{1}{2}\int_{t_0}^{t} \dot{z}_2(\tau)^2 d\tau} e^{-\frac{1}{2}\omega^2 \int_{t_0}^{t} z_2(\tau)^2 d\tau - \frac{1}{2}(\lambda^2 - \frac{1}{4}) \int_{t_0}^{t} z_2(\tau)^2 d\tau} \delta\big(z_2(t_0)^2 - z_{20}^2\big)\, dk. \tag{41}$$

Here the following was denoted:

$$\omega = \frac{1}{4}\sqrt{4\kappa^2 + (4k^2+1)\sigma^2}, \quad \lambda = \alpha - 1,$$
$$z_2 = \frac{2}{\sigma}\sqrt{V}, \quad z_{20} = \frac{2}{\sigma}\sqrt{V_0}.$$

The path integral in (41) is also known as integral for radial oscillator [17] and it's value is given in Appendix C. By substituting value of integral (50) into formula (41) and after a number of transformations we receive the following

$$K(S, S_0, V, V_0, t, t_0) = \frac{2}{\sigma^2} e^{\frac{1}{2}(\mu+\alpha\kappa)(t-t_0)} \sqrt{\frac{S_0}{S^3}} e^{-\frac{(V-V_0)\kappa}{\sigma^2}} \left(\frac{V}{V_0} \right)^{\frac{1}{2}(\alpha-1)} \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{\mathrm{i}\, k(\ln(S/S_0)-\mu(t-t_0))}$$
$$\times \exp\left(- \frac{2(V+V_0)\omega \coth(\omega(t-t_0))}{\sigma^2} \right) \frac{\omega}{\sinh(\omega(t-t_0))} I_\lambda\left(\frac{4\sqrt{VV_0}\,\omega}{\sigma^2 \sinh(\omega(t-t_0))} \right) dk. \tag{42}$$

This way formula (42) gives solution for transitional probability density in Heston model. As we know, there was no such solution given in the literature before. As we already noted the FP equation for Heston model is solved in work of [20], where there was the Laplace transform over variable V used. However there was no inverse Laplace transform given for the solution. For that matter let us consider transitional probability density for stochastic variable S for $\forall V \in [0, \infty]$

$$K(S, S_0, V_0, t, t_0) = \int_0^{\infty} K(S, S_0, V, V_0, t, t_0)\, dV \tag{43}$$

and compare it to the respective expression from work [20]. After integrating (42) over variable V we receive

$$K(S, S_0, V_0, t, t_0) = e^{\frac{1}{2}(\mu+\alpha\kappa)(t-t_0)} \sqrt{\frac{S_0}{S^3}} \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{\mathrm{i}\, k(\ln(S/S_0)-\mu(t-t_0))}$$
$$\times \exp\left(- \frac{(1+4k^2)V_0}{4(\kappa + 2\omega\coth(\omega(t-t_0)))} \right) \left(\cosh(\omega(t-t_0)) + \frac{\kappa \sinh(\omega(t-t_0))}{2\omega} \right)^{-\alpha} dk. \tag{44}$$

Also in (44) we shall perform variable substitution $x = \ln \frac{S}{S_0}$ and make a drift of variable k in complex plane $k \to k - \frac{\mathrm{i}}{2}$. This transformation is reasonable since the integral function (44) is analytic in the region $-\frac{1}{2}\sqrt{1+(2\kappa/\sigma)^2} < k_s < \frac{1}{2}\sqrt{1+(2\kappa/\sigma)^2}$ of complex plane ($k = k_c + \mathrm{i}\, k_s$). As a result we shall write the following

$$K(x, V_0, t, t_0) = e^{\frac{1}{2}\alpha\kappa(t-t_0)} \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{\mathrm{i}\, k(x-\mu(t-t_0))} \times \exp\left(-\frac{V_0 k(k-\mathrm{i})}{\kappa + 2\tilde{\omega}\coth(\tilde{\omega}(t-t_0))}\right)\left(\cosh(\tilde{\omega}(t-t_0)) + \frac{\kappa \sinh(\tilde{\omega}(t-t_0))}{2\tilde{\omega}}\right)^{-\alpha} dk, \quad (45)$$

where the following is denoted

$$\tilde{\omega} = \frac{1}{2}\sqrt{\kappa^2 + k(k-\mathrm{i})\sigma^2}.$$

Expression (45) for transitional probability density matches with the one given in [20]. Using formula (45) it can be seen that transitional probability density normalized $\int_{-\infty}^{\infty} K(x, V_0, t, t_0)\, dx = 1$.

By a similar approach one can obtain transitional probability density for stochastic variable V by integrating over S in (42)

$$K(V, V_0, t, t_0) = e^{\frac{1}{2}\alpha\kappa(t-t_0)} e^{-\frac{(V-V_0)\kappa}{\sigma^2}} \left(\frac{V}{V_0}\right)^{\frac{1}{2}(\alpha-1)} \times \exp\left(-\frac{(V+V_0)\kappa\coth(\frac{1}{2}\kappa(t-t_0))}{\sigma^2}\right) \frac{\kappa}{\sigma^2 \sinh\frac{1}{2}\kappa(t-t_0)}\, I_\lambda\left(\frac{2\sqrt{VV_0}\,\kappa}{\sigma^2\sinh\frac{1}{2}\kappa(t-t_0)}\right). \quad (46)$$

The received expression (46) is a known solution to the Cox–Ingersoll–Ross model [21] (the second equation of (29)) for the transitional probability density.

Let us consider some peculiarities that arise as a result of application of path integrals to solution of FP equation in a number of works we mentioned earlier. In particular in [16] the FP equation received based on a system of SDE in Stratonovich scheme was considered. As it is known from [12, 15, 16] in order to transition to Stratonovich scheme in (23) one should perform variable substitution

$$A_i(x) \to A_i(x) + \frac{1}{2}\sum_{k,j=1}^{n} B_{jk}\frac{\partial B_{ik}}{\partial x_j}.$$

With all that being said let us find out a difference between Lagrange functions in [16] (L_{AR}) for Heston model and the expression found based on (32) $L(\tau)$

$$L(\tau) - L_{AR} = \frac{\dot{S}(\tau)}{2S(\tau)} - \frac{V(\tau)}{8} + \frac{\dot{V}(\tau) + \theta\kappa - \frac{\sigma^2}{4}}{4V(\tau)} - \frac{\kappa}{4}.$$

In work [12] for a multi-dimensional FP equation the path integral was used for propagator of Schrödinger equation in a curved space of imaginary time. Comparing $L(\tau)$ with Lagrange function (L_{RC}) [12] for Heston model we find that

$$L(\tau) - L_{RC} = \frac{\sigma^2}{96V(\tau)}.$$

One can show that "scalar curvature" [12] for Heston model is equal to $R = \frac{\sigma^2}{V(\tau)}$. It follows that the other options of path integral where the Lagrange functions differs by a multiplier near the term with a scalar curvature [14, 15] lead to a different results than those from this work. In [15] a path integrals with Lagrange functions are given, which apart from other multiplier near the scalar curvature term contain other terms of a fairly complicated structure. However we do not give comparison them in this case. Regarding the MSRJD method for building the propagator of Schrödinger equation an it's application to the FP equation solutions, the respective formulas in [15] contain double path integrals (over conjugated variables) and so it is impossible to perform comparison of Lagrange function.

3.2. Correlated Wieners processes $\rho \neq 0$

In the case of correlated Wieners processes the algorithm of obtaining a solutions follows the one described above, hence we give the summary result. Transitional probability density is equal to (for variable $x = \ln \frac{S}{S_0}$)

$$K(x,V,t) = \frac{2}{\sigma^2}\left(\frac{V}{V_0}\right)^{\frac{1}{2}(\alpha-1)} \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{\mathrm{i}\,k(x-\mu(t-t_0))} e^{\frac{1}{2}\alpha\gamma(t-t_0)} e^{-\frac{\gamma(V-V_0)}{\sigma^2}} \\ \times \exp\left(-\frac{2(V+V_0)\,\omega\coth(\omega(t-t_0))}{\sigma^2}\right)\frac{\omega}{\sinh\big(\omega(t-t_0)\big)}\, I_{\alpha-1}\left(\frac{4\sqrt{V_0V}\omega}{\sigma^2\sinh\big(\omega(t-t_0)\big)}\right)dk. \quad (47)$$

Following is denoted:

$$\gamma = \kappa + \mathrm{i}\,k\rho\,\sigma, \quad \omega = \frac{1}{2}\sqrt{\gamma^2 + k(k-\mathrm{i})\sigma^2}.$$

Integrating over variable V in (47) we receive transitional probability density for stochastic variable x

$$K(x,t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{\mathrm{i}\,k(x-\mu(t-t_0))} e^{\frac{1}{2}\alpha\gamma(t-t_0)} \exp\left(-\frac{k(k-\mathrm{i})V_0}{\gamma + 2\omega\coth(\omega(t-t_0))}\right) \\ \times\left(\cosh(\omega(t-t_0)) + \frac{\gamma\sinh(\omega(t-t_0))}{2\omega}\right)^{-\alpha} dk. \quad (48)$$

From formula (48) it also follows, that $K(x,t)$ is normalized and formula (48) transforms into (45) for $\rho = 0$. The expression for transitional probability density (48) matches with an analogous formula given in [20]. From formula (45), (48) it can be seen that they define a ground expression for characteristic function of Heston model of variable $x = \ln \frac{S}{S_0}$, since integral over k is fairly complicated.

4. Conclusions

The path integral method was applied to solution of FP equation of multiple variables that corresponds to a system of SDE. Solution for transitional probability density of stochastic variables written in the form of path integral is given. The solutions for transitional probability density in Heston model was found in [20] based on FP equation using integral transforms of Laplace and Fourier over corresponding variables. As a result it is shown that the solutions obtained using path integrals matches the one given in [20] work.

Based on an example of Heston model it is also shown that path integrals obtained in a number of works for FP equation based on quantum-mechanical analogy [12–15] lead to different results. We illustrated that application of a known quantum-mechanical approach for FP equation does not allow to receive the path integral given in the current work.

The path integral for transitional probability density is convenient for analyzing models of multiple variables that contain two or more SDE [22, 23]. Those and others problems will be a subject of study of a separate work.

Appendix A

Specified Jacobian of variable substitution (18), (19) is equal to functional determinant of derivative matrix

$$J = \left\|\frac{\delta q_k(\tau)}{\delta\nu_i(\tau')}\right\|.$$

As a result we receive

$$J = \left\|B_{ki}^{-1}(x(\tau))\,\delta(\tau-\tau') - \sum_{j}^{n}\frac{\partial B_{kj}^{-1}(x(\tau))}{\partial x_i}\,\theta(\tau'-\tau)\,\nu_j(\tau)\right\|.$$

After a number of transformations we receive

$$J = \left(\prod_{\tau=t_0}^{t}\frac{1}{\det(B(x(\tau)))}\right)\left\|\delta_{kk'}\,\delta(\tau-\tau') + \theta(\tau'-\tau)\sum_{ij}^{n}\frac{\partial B_{k'j}(x(\tau))}{\partial x_k}\,B_{ji}^{-1}(x(\tau))\,\nu_i(\tau)\right\|.$$

As a result, we receive the following for Jacobian [11, 18]

$$J = \left(\prod_{\tau=t_0}^{t}\frac{1}{\det(B(x(\tau)))}\right)\exp\left(\frac{1}{2}\sum_{kij}^{n}\int_{t_0}^{t}\frac{\partial B_{kj}(x(\tau))}{\partial x_k}B_{ji}^{-1}(x(\tau))\,\nu_i(\tau)\,d\tau\right). \quad (49)$$

Appendix B

Let us consider the Fourier integral

$$\delta(Se^x - S_0) = \frac{1}{2\pi}\int_{-\infty}^{\infty} f(k)\, e^{\mathrm{i}\,kx} dk.$$

From where we receive $f(k)$

$$f(k) = \int_{-\infty}^{\infty} \delta(Se^x - S_0)\, e^{-\mathrm{i}\,kx} dx = \frac{1}{S_0} e^{-\mathrm{i}\,k \ln(S_0/S)}.$$

As a result

$$\delta(Se^x - S_0) = \frac{1}{2\pi S_0}\int_{-\infty}^{\infty} e^{\mathrm{i}\,k(x - \ln(S_0/S))} dk.$$

Appendix C

The path integral in formula (41) is known from the problem of radial oscillator and has analytical solution [7,17]. Transforming to the "velocity" variables we receive

$$\int \mathcal{D}q(\tau) \exp\left(-\frac{1}{2}\int_{t_0}^{t} \dot{z}(\tau)^2 d\tau\right) \exp\left(-\frac{1}{2}\omega^2 \int_{t_0}^{t} z(\tau)^2 d\tau - \frac{1}{2}\left(\lambda^2 - \frac{1}{4}\right)\int_{t_0}^{t} \frac{d\tau}{z(\tau)^2}\right) \delta\big(z(t_0)^2 - z_0^2\big)$$
$$= \frac{1}{2}\exp\left(-\frac{1}{2}(z^2 + z_0^2)\omega \coth(\omega(t - t_0))\right)\sqrt{\frac{z}{z_0}}\frac{\omega}{\sinh(\omega(t - t_0))} I_\lambda\left(\frac{z z_0\, \omega}{\sinh(\omega(t - t_0))}\right). \quad (50)$$

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Розв'язок рівняння Фоккера–Планка в методі функціонального інтегрування

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Розглянуто рівняння Фоккера–Планка багатьох змінних, що відповідає системі СДР. Розв'язок для густини умовної ймовірності записаний у вигляді функціонального інтегралу. Показано, що для моделі Гестона запропонований функціональний інтеграл приводить для відомого результату отриманого іншим шляхом. Вказано також на відмінності результатів на основі функціональних інтегралів наведених у ряді робіт.

Ключові слова: *стохастичні диференціальні рівняння; рівняння Фоккера–Планка; умовна ймовірність, функціональний інтеграл; модель Гестона.*