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Numerical studies of a Timoshenko system with the second sound

Smouk A., Radid A.

Department of Mathematics and Informatics, Hassan II University, FSAC, Fundamental and Applied Mathematics Laboratory, Casablanca, Morocco

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Timoshenko's problem is not a recent problem and many articles exist concerning his study. New physical problems appear and require a good mathematical understanding of the behavior of this phenomenon. Our contribution will consist in studying the numerical stability of a Timoshenko system with second sound. We introduce a finite element approximation and prove that the associated discrete energy decreases and we establish a priori error estimates. Finally, some numerical simulations are obtained.

Keywords: *Timoshenko's problem; numerical stability; finite element method; numerical simulations.*

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1. Introduction

Timoshenko [1], was one who initially unveiled the form-based

$$\begin{cases} \rho\,\omega_{tt} - k(\omega_x + \lambda)_x = 0, & (x,t) \in (0,L) \times \mathbb{R}^+, \\ I_\rho\,\lambda_{tt} - (EI\lambda_x)_x + k(\omega_x + \lambda) = 0, & (x,t) \in (0,L) \times \mathbb{R}^+, \end{cases} \tag{1}$$

where ω is the transverse displacement of the beam, λ is the rotation angle of the filament of the beam. The constants ρ, I_ρ, E, I and k reflect the density (the mass per unit length), the polar moment of inertia of a cross section, Young's modulus of elasticity, the moment of inertia of a cross section, and the shear modulus, respectively. In this system the coefficients $k = k_1 GA$, where G and k_1 are respectively the modulus of rigidity and the transverse factor. There are several publications related to the study of Timoshenko systems [2–4]. The stability of these systems has attracted considerable attention in recent years, and a number of deductions relating uniform and asymptotic energy decay have been developed [1, 5, 6].

Fernández and Rake [7] studied the following related of two Timoshenko type wave equations with heat conduction

$$\begin{cases} \rho_1\omega_{tt} - k(\omega_x + \lambda)_x = 0, \\ \rho_2\lambda_{tt} - b\lambda_{xx} + k(\omega_x + \lambda) + \gamma\vartheta_x = 0, \\ \rho_3\vartheta_t + q_x + \gamma\lambda_{xt} = 0, \\ \tau q_t + \beta q + \vartheta_x = 0, \end{cases} \tag{2}$$

and proved that, in the absence of additional frictional damping, coupling via Cattaneo's law results in the loss of the exponential decay typically achieved through coupling by Fourier's law [8].

Santos et al. [9] considered Timoshenko's beam model with the second sound and introduced a stability number μ defined by:

$$\mu = \left(\tau - \frac{\rho_1}{k\rho_3}\right)\left(\frac{\rho_2}{b} - \frac{\rho_1}{k}\right) - \frac{\tau\delta^2\rho_1}{bk\rho_3},$$

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which characterizes exponential decay. Santos et al. showed that the corresponding semigroup associated with the system (2) is exponentially stable if and only if $\mu = 0$ and a polynomial decay for $\mu \neq 0$.

In this paper, our focus is to the Timoshenko system:

$$\begin{cases} \rho_1\omega_{tt} - k(\omega_x + \lambda)_x = 0, \\ \rho_2\lambda_{tt} - b\lambda_{xx} + k(\omega_x + \lambda) + \gamma\vartheta_x + \sigma(t)f(\lambda_t) = 0, \\ \rho_3\vartheta_t + q_x + \gamma\lambda_{xt} = 0, \\ \tau q_t + \beta q + \vartheta_x = 0, \end{cases} \tag{3}$$

here t is the time, x is the position coordinate along the beam, the functions ω, λ, ϑ, q indicate respectively the transverse displacement of a curved beam, the rotation angle of the filament of the beam, the temperature and the heat flux, $(x,t) \in (0,L)\times(0,\infty)$ with L represents the distance between the ends of the center line of the beam and ρ_1, ρ_2, ρ_3, b, k, γ, β are positive coefficients. Also, σ and f verify the assumptions:

(i) $\sigma\colon \mathbb{R}_+ \to \mathbb{R}_+$ is a decreasing differentiable function.

(ii) $f\colon \mathbb{R} \to \mathbb{R}$ is a locally Lipschitz function satisfying $f(0) = 0$.

(iii) $f\colon \mathbb{R} \to \mathbb{R}$ is a continuous non-decreasing function with $f(0) = 0$ and there exists a continuous strictly increasing odd function $f_0 \in C([0,+\infty))$, continuously differentiable in a neighborhood of 0, satisfying $f_0(0) = 0$ and such that

$$\begin{cases} f_0(z) \leqslant |f(z)| \leqslant f_0^{-1}(z), & \text{for all } |z| \leqslant \varepsilon, \\ b_1|z| \leqslant |f(z)| \leqslant b_2|z|, & \text{for all } |z| \geqslant \varepsilon, \end{cases}$$

b_1 and b_2 are positive.

Additionally, we construct a function F by

$$F(x) = \sqrt{x}\, f_0\left(\sqrt{x}\right).$$

In light of assumption (iii), F is of class C^1 and is strictly convex on $(0, \varrho^2]$, where $\varrho > 0$ is a sufficiently small number.

With (3), we associate the boundary conditions given by

$$\omega_x(0,t) = \omega_x(1,t) = \lambda(0,t) = \lambda(1,t) = q(0,t) = q(1,t) = 0, \quad \forall t \geqslant 0, \tag{4}$$

and the following initial conditions

$$\begin{cases} \omega(x,0) = \omega_0(x), & \omega_t(x,0) = \omega_1(x), & \forall x \in (0,1), \\ \lambda(x,0) = \lambda_0(x), & \lambda_t(x,0) = \lambda_1(x), & \forall x \in (0,1), \\ \vartheta(x,0) = \vartheta_0(x), & q(x,0) = q_0(x), & \forall x \in (0,1). \end{cases} \tag{5}$$

Remark 1. Hypothesis (iii) was introduced by Lasiecka and Tataru [10] implies that $z\,f(z) > 0$, for all $z \neq 0$.

Under the hypothesis (i), (ii) and (iii) Ayadi et al. [11] established specific and common results for a large class of relaxating functions whose stability number μ is dependent. The system's energy (3) is determined by:

$$E(t) = \frac{1}{2}\int_0^L \left(\rho_1\omega_t^2 + \rho_2\lambda_t^2 + b\lambda_x^2 + k(\omega_x + \lambda)^2 + \rho_3\vartheta^2 + \tau q^2\right) dx$$

satisfies

$$\frac{dE}{dt} = -\beta\int_0^1 q^2 dx - \alpha(t)\int_0^1 \lambda_t f(\lambda_t)\, dx.$$

Our contribution consists of introducing a FE (finite element) approximation and we prove that the associated discrete energy decreases over time with different values of μ. In addition, we also get a priori error estimates. Finally, we show that the numerical results are consistent with our theoretical results [11].

2. Numerical analysis

We provide in this part a FE approximation to system (3) with the boundary conditions (4) and the initial conditions (5). Also, we present and study an implicit Euler type scheme based on finite differences in time and FE en space. We prove that the discrete energy decays.

2.1. Stability of the scheme

We consider the following functions $\widetilde{\omega} = \omega_t$, $\widetilde{\lambda} = \lambda_t$ and the system (3) is rewritten as follows:

$$\begin{cases} \rho_1 \widetilde{\omega}_t - k(\omega_x + \lambda)_x = 0, \\ \rho_2 \widetilde{\lambda}_t - b\lambda_{xx} + k(\omega_x + \lambda) + \gamma \vartheta_x + \sigma(t) f(\widetilde{\lambda}) = 0, \\ \rho_3 \vartheta_t + q_x + \gamma \widetilde{\lambda}_x = 0, \\ \tau q_t + \beta q + \vartheta_x = 0. \end{cases} \tag{6}$$

To get the weak form associated to system (6), we multiply the equations by test functions ζ, χ, η, $\alpha \in H^1(0,1)$ and integrating by parts,

$$\begin{cases} \rho_1 \big(\widetilde{\omega}_t, \zeta\big) + k\left(\omega_x + \lambda, \zeta_x\right) = 0, \\ \rho_2 \big(\widetilde{\lambda}_t, \chi\big) + b\left(\lambda_x, \chi_x\right) + k\left(\omega_x + \lambda, \chi\right) + \gamma\left(\vartheta_x, \chi\right) + \sigma(t)\big(f(\widetilde{\lambda}), \chi\big) = 0, \\ \rho_3 \left(\vartheta_t, \eta\right) + \left(q_x, \eta\right) + \gamma\big(\widetilde{\lambda}_x, \eta\big) = 0, \\ \tau\left(q_t, \alpha\right) + \beta\left(q, \alpha\right) + \left(\vartheta_x, \alpha\right) = 0. \end{cases} \tag{7}$$

For our purposes, referred to J as a nonnegative integer and $h = \frac{1}{J}$ a subdivision of the interval $(0,1)$ given by $0 = x_0 < x_1 < \ldots < x_{J-1} < x_J = 1$, such that $x_j = jh$, $\forall j = 0, \ldots, J$ and

$$S^h = \big\{u \in H^1(0,1) | u \in C([0,1]), u|_{(x_j, x_{j+1})} \text{ is a linear polynomial, with } j = 0, \ldots, J-1\big\} \tag{8}$$

and

$$S_0^h = \big\{u \in S^h | u(0) = u(1) = 0\big\}.$$

Given a certain final time T and a positive integer N, have $\Delta t = T/N$ be the time step and $t_n = n\Delta t$, $n = 0, \ldots, N$.

The FE method for (7) with Dirichlet homogeneous boundary conditions using the backward Euler scheme is to find $\widetilde{\omega}_h^n \in S^h$, $\widetilde{\lambda}_h^n$, ϑ_h^n and $q_h^n \in S_0^h$ such that, for $n = 1, \ldots, N$ and for all ζ_h, χ_h, η_h, $\alpha_h \in S^h$

$$\begin{cases} \frac{\rho_1}{\Delta t}\left(\widetilde{\omega}_h^n - \widetilde{\omega}_h^{n-1}, \zeta_h\right) + k\left(\omega_{hx}^n + \lambda_h^n, \zeta_{hx}\right) = 0, \\ \frac{\rho_2}{\Delta t}\big(\widetilde{\lambda}_h^n - \widetilde{\lambda}_h^{n-1}, \chi_h\big) + b\big(\lambda_{hx}^n, \chi_{hx}\big) + k\left(\omega_{hx}^n + \lambda_h^n, \chi_h\right) + \gamma\left(\vartheta_{hx}^n, \chi_h\right) + \sigma^n\big(f(\widetilde{\lambda_h^n}), \chi_h\big) = 0, \\ \frac{\rho_3}{\Delta t}\left(\vartheta_h^n - \vartheta_h^{n-1}, \eta_h\right) + \left(q_{hx}^n, \eta_h\right) + \gamma\big(\widetilde{\lambda}_x^n, \eta_h\big) = 0, \\ \frac{\tau}{\Delta t}\left(q_h^n - q_h^{n-1}, \alpha_h\right) + \beta\left(q_h^n, \alpha_h\right) + \left(\vartheta_{hx}^n, \alpha_h\right) = 0, \end{cases} \tag{9}$$

where

$$\widetilde{\omega}_h^n = \frac{\omega_h^n - \omega_h^{n-1}}{\Delta t}, \quad \widetilde{\lambda}_h^n = \frac{\lambda_h^n - \lambda_h^{n-1}}{\Delta t} \tag{10}$$

are approximate representations of $\omega_t(t_n)$, $\lambda_t(t_n)$ respectively.

For the initial conditions ω_0, ω_1, λ_0, λ_1, ϑ_0, q_0 are given by ω_h^0, $\widetilde{\omega}_h^0$, λ_h^0, $\widetilde{\lambda}_h^0$, ϑ_h^0 and q_h^0 respectively.

A discrete expression for the energy decay property met by the solution of system (3) is the next outcome.

Theorem 1. *The discrete energy can be:*

$$\mathcal{E}_h^n = \frac{1}{2}\left(\rho_1\big\|\widetilde{\omega}_h^n\big\|^2 + \rho_2\big\|\widetilde{\lambda}_h^n\big\|^2 + k\big\|\omega_{hx}^n + \lambda_h^n\big\|^2 + b\big\|\lambda_{hx}^n\big\|^2 + \rho_3\big\|\vartheta_h^n\big\|^2 + \tau\big\|q_h^n\big\|^2\right). \tag{11}$$

Then, the decay property as follow

$$\frac{\mathcal{E}_h^n - \mathcal{E}_h^{n-1}}{\triangle t} \leqslant 0, \tag{12}$$

holds for $n = 1, 2, \ldots, N$, where $\|\cdot\|$ is the norm of $L^2(0,1)$.

Proof. The next inequality is utilized frequently:

$$(u_1 - u_2, u_1) = \frac{1}{2}\left(\|u_1 - u_2\|^2 + \|u_1\|^2 - \|u_2\|^2\right). \tag{13}$$

Taking $\zeta_h = \widetilde{\omega}_h^n$, $\chi_h = \widetilde{\lambda}_h^n$, $\eta_h = \vartheta_h^n$ and $\alpha_h = q_h^n$ in (9).

Recalling (10) and (13), we deduce that

$$\frac{\rho_1}{2\Delta t}\left(\|\widetilde{\omega}_h^n - \widetilde{\omega}_h^{n-1}\|^2 + \|\widetilde{\omega}_h^n\|^2 - \|\widetilde{\omega}_h^{n-1}\|^2\right) + k\left(\omega_{hx}^n + \lambda_h^n, \widetilde{\omega}_{hx}^n\right) = 0, \tag{14}$$

$$\begin{aligned}&\frac{\rho_2}{2\Delta t}\left(\|\widetilde{\lambda}_h^n - \widetilde{\lambda}_h^{n-1}\|^2 + \|\widetilde{\lambda}_h^n\|^2 - \|\widetilde{\lambda}_h^{n-1}\|^2\right) + \frac{b}{2\Delta t}\left(\|\lambda_{hx}^n - \lambda_{hx}^{n-1}\|^2 + \|\lambda_{hx}^n\|^2 - \|\lambda_{hx}^{n-1}\|^2\right)\\ &\quad + k\big(\omega_{hx}^n + \lambda_h^n, \widetilde{\lambda}_h^n\big) + \gamma\big(\vartheta_{hx}^n, \widetilde{\lambda}_h^n\big) + \sigma^n\big(f(\widetilde{\lambda_h^n}), \widetilde{\lambda}_h^n\big) = 0,\end{aligned} \tag{15}$$

$$\frac{\rho_3}{2\Delta t}\left(\|\vartheta_h^n - \vartheta_h^{n-1}\|^2 + \|\vartheta_h^n\|^2 - \|\vartheta_h^{n-1}\|^2\right) - \left(q_h^n, \vartheta_{hx}^n\right) - \gamma\big(\widetilde{\lambda}_h^n, \vartheta_{hx}^n\big) = 0, \tag{16}$$

$$\frac{\tau}{2\Delta t}\left(\|q_h^n - q_h^{n-1}\|^2 + \|q_h^n\|^2 - \|q_h^{n-1}\|^2\right) + \beta\|q_h^n\|^2 + \big(\vartheta_{hx}^n, q_h^n\big) = 0. \tag{17}$$

Using again (10) and (13),

$$\begin{aligned}(u^n, \widetilde{u}^n) &= \left(u^n, \frac{u^n - u^{n-1}}{\Delta t}\right),\\ &= \frac{1}{2\Delta t}\left(\|u^n - u^{n-1}\|^2 + \|u^n\|^2 - \|u^{n-1}\|^2\right),\\ &\geqslant \frac{1}{2\Delta t}\left(\|u^n\|^2 - \|u^{n-1}\|^2\right),\end{aligned} \tag{18}$$

Results as:

$$k\big(\omega_{hx}^n + \lambda_h^n, \widetilde{\omega}_{hx}^n + \widetilde{\lambda}_h^n\big) \geqslant \frac{k}{2\Delta t}\left(\|\omega_{hx}^n + \lambda_h^n\|^2 - \|\omega_{hx}^{n-1} + \lambda_h^{n-1}\|^2\right), \tag{19}$$

and

$$\sigma^n\big(f(\widetilde{\lambda_h^n}), \widetilde{\lambda}_h^n\big) \geqslant 0$$

summing equations (14)–(17),

$$\begin{aligned}0 &= \frac{\rho_1}{2\Delta t}\left(\|\widetilde{\omega}_h^n\|^2 - \|\widetilde{\omega}_h^{n-1}\|^2\right) + \frac{\rho_2}{2\Delta t}\left(\|\widetilde{\lambda}_h^n\|^2 - \|\widetilde{\lambda}_h^{n-1}\|^2\right)\\ &\quad + \frac{\rho_3}{2\Delta t}\left(\|\vartheta_h^n\|^2 - \|\vartheta_h^{n-1}\|^2\right) + \frac{\tau}{2\Delta t}\left(\|q_h^n\|^2 - \|q_h^{n-1}\|^2\right)\\ &\quad + \frac{b}{2\Delta t}\left(\|\lambda_{hx}^n\|^2 - \|\lambda_{hx}^{n-1}\|^2\right) + \frac{\rho_1}{2\Delta t}\|\widetilde{\omega}_h^n - \widetilde{\omega}_h^{n-1}\|^2 + \frac{\rho_2}{2\Delta t}\|\widetilde{\lambda}_h^n - \widetilde{\lambda}_h^{n-1}\|^2\\ &\quad + \frac{\tau}{2\Delta t}\|q_h^n - q_h^{n-1}\|^2 + \frac{b}{2\Delta t}\|\lambda_{hx}^n - \lambda_{hx}^{n-1}\|^2 + \frac{\rho_3}{2\Delta t}\|\vartheta_h^n - \vartheta_h^{n-1}\|^2\\ &\quad + \beta\|q_h^n\|^2 + k\big(\omega_{hx}^n + \lambda_h^n, \widetilde{\omega}_{hx}^n + \widetilde{\lambda}_h^n\big) + \sigma^n\big(f(\widetilde{\lambda_h^n}), \widetilde{\lambda}_h^n\big)\\ &\geqslant \frac{\rho_1}{2\Delta t}\left(\|\widetilde{\omega}_h^n\|^2 - \|\widetilde{\omega}_h^{n-1}\|^2\right) + \frac{\rho_2}{2\Delta t}\left(\|\widetilde{\lambda}_h^n\|^2 - \|\widetilde{\lambda}_h^{n-1}\|^2\right) + \frac{b}{2\Delta t}\left(\|\lambda_{hx}^n\|^2 - \|\lambda_{hx}^{n-1}\|^2\right)\\ &\quad + \frac{\rho_3}{2\Delta t}\left(\|\vartheta_h^n\|^2 - \|\vartheta_h^{n-1}\|^2\right) + \frac{\tau}{2\Delta t}\left(\|q_h^n\|^2 - \|q_h^{n-1}\|^2\right)\\ &\quad + \frac{k}{2\Delta t}\left(\|\omega_{hx}^n + \lambda_h^n\|^2 - \|\omega_{hx}^{n-1} + \lambda_h^{n-1}\|^2\right)\\ &= \frac{\mathcal{E}_h^n - \mathcal{E}_h^{n-1}}{\Delta t}.\end{aligned} \tag{20}$$

As a result, we have $\frac{\mathcal{E}_h^n - \mathcal{E}_h^{n-1}}{\Delta t} \leqslant 0$. ■

As an outcome, the next stability estimations are obtained.

Remark 2. The solution of (9) can be found by solving a square linear system of algebraic equations. According to the above demonstration, when all data are zero, the solution $\left\{\widetilde{\omega}_h^n, \widetilde{\lambda}_h^n, \vartheta_h^n, q_h^n\right\}$ is zero. Following which, the solution uniqueness of (9).

2.2. Error estimate

In the present part, we derive a priori error estimates on the numerical approximation of the Timoshenko system with the second sound (3) in the case of $\sigma(t) = 1$ and $f(z) = z$, in which we obtain the convergence of the error (see Figures 9 and 10).

Theorem 2. *There is* $C > 0$, *independent of the parameters* h *and* Δt *such that* $\forall \{\zeta_h^i, \chi_h^i, \eta_h^i, \alpha_h^i\}_{i=0}^N \in S^h$

$$\begin{aligned}
&\|\widetilde{\omega}^n - \widetilde{\omega}_h^n\|^2 + \|\widetilde{\lambda}^n - \widetilde{\lambda}_h^n\|^2 + \|\widetilde{\lambda}_x^n - \widetilde{\lambda}_{hx}^n\|^2 + \|\omega_x^n + \lambda^n - (\omega_{hx}^n + \lambda_h^n)\|^2 + \|\vartheta^n - \vartheta_h^n\|^2 + \|q^n - q_h^n\|^2 \\
&\leqslant C\Delta t \sum_{i=1}^N \left\|\widetilde{\omega}_t^i - \frac{\widetilde{\omega}^i - \widetilde{\omega}^{i-1}}{\Delta t}\right\|^2 + \left\|\widetilde{\lambda}_t^i - \frac{\widetilde{\lambda}^i - \widetilde{\lambda}^{i-1}}{\Delta t}\right\|^2 + \left\|\vartheta_t^i - \frac{\vartheta^i - \vartheta^{i-1}}{\Delta t}\right\|^2 + \left\|q_t^i - \frac{q^i - q^{i-1}}{\Delta t}\right\|^2 \\
&+ \|\widetilde{\omega}^i - \zeta_h^i\|^2 + \|\widetilde{\omega}_x^i - \zeta_{hx}^i\|^2 + \|\widetilde{\lambda}^i - \chi_h^i\|^2 + \|\widetilde{\lambda}_x^i - \chi_{hx}^i\|^2 + \|\vartheta_x^i - \eta_{hx}^i\|^2 + \|q_x^i - \alpha_{hx}^i\|^2 + \|\vartheta^i - \eta_h^i\|^2 \\
&+ \|q^i - \alpha_h^i\|^2 + \frac{C}{\Delta t}\sum_{i=1}^{N-1}\Big(\|\widetilde{\omega}^i - \zeta_h^i - (\widetilde{\omega}^{i+1} - \zeta_h^{i+1})\|^2 + \|\widetilde{\lambda}^i - \chi_h^i - (\widetilde{\lambda}^{i+1} - \chi_h^{i+1})\|^2 \\
&+ \|\vartheta^i - \eta_h^i - (\vartheta^{i+1} - \eta_h^{i+1})\|^2 + \|q^i - \alpha_h^i - (q^{i+1} - \alpha_h^{i+1})\|^2\Big) + C\Big(\|\omega^1 - \widetilde{\omega}_h^0\|^2 + \|\lambda^1 - \widetilde{\lambda}_h^0\|^2 \\
&+ \|\lambda_x^0 - \lambda_{hx}^0\|^2 + \|\vartheta^0 - \vartheta_h^0\|^2 + \|q^0 - q_h^0\|^2 + \|\omega_{hx}^0 + \lambda_h^0 - (\omega_x^0 + \lambda^0)\|^2\Big). \qquad (21)
\end{aligned}$$

Proof. Step 1: For a continuous function $g(t)$, let $g^n = g(t_n)$. Subtracting equation $(7)_1$ at time t_n for $\zeta = \zeta_h \in S^h$ and the discrete variational equation $(9)_1$, we obtain

$$\rho_1\left(\widetilde{\omega}_t^n - \frac{\widetilde{\omega}_h^n - \widetilde{\omega}_h^{n-1}}{\Delta t}, \zeta_h\right) + k\big((\omega_x^n + \lambda^n) - (\omega_{hx}^n + \lambda_h^n), \zeta_{hx}\big) = 0. \qquad (22)$$

Thus, for all $\zeta \in S^h$,

$$\begin{aligned}
&\rho_1\left(\widetilde{\omega}_t^n - \frac{\widetilde{\omega}_h^n - \widetilde{\omega}_h^{n-1}}{\Delta t}, \widetilde{\omega}^n - \widetilde{\omega}_h^n\right) + k\big(\omega_x^n + \lambda^n - (\omega_{hx}^n + \lambda_h^n), \widetilde{\omega}_x^n - \widetilde{\omega}_{hx}^n\big) \\
&= \rho_1\left(\widetilde{\omega}_t^n - \frac{\widetilde{\omega}_h^n - \widetilde{\omega}_h^{n-1}}{\Delta t}, \widetilde{\omega}^n - \zeta_h\right) + k\big(\omega_x^n + \lambda^n - (\omega_{hx}^n + \lambda_h^n), \widetilde{\omega}_x^n - \zeta_{hx}\big). \qquad (23)
\end{aligned}$$

Similarly, from equations $(7)_2$–$(7)_4$ and $(9)_2$–$(9)_4$ we deduce, for all $\chi_h, \xi_h, \eta_h, \alpha_h \in S^h$

$$\begin{aligned}
&\rho_2\left(\widetilde{\lambda}_t^n - \frac{\widetilde{\lambda}_h^n - \widetilde{\lambda}_h^{n-1}}{\Delta t}, \widetilde{\lambda}^n - \widetilde{\lambda}_h^n\right) + b\big(\lambda_x^n - \lambda_{hx}^n, \widetilde{\lambda}_x^n - \widetilde{\lambda}_{hx}^n\big) \\
&+ k\big(\omega_x^n + \lambda^n - (\omega_{hx}^n + \lambda_h^n)\widetilde{\lambda}^n - \widetilde{\lambda}_h^n\big) - \gamma\big(\vartheta^n - \vartheta_h^n, \widetilde{\lambda}_x^n - \widetilde{\lambda}_{hx}^n\big) + \big(\widetilde{\lambda}^n - \widetilde{\lambda}_h^n, \widetilde{\lambda}^n - \widetilde{\lambda}_h^n\big) \\
&= \rho_2\left(\widetilde{\lambda}_t^n - \frac{\widetilde{\lambda}_h^n - \widetilde{\lambda}_h^{n-1}}{\Delta t}, \widetilde{\lambda}^n - \chi_h\right) + b\big(\lambda_x^n - \lambda_{hx}^n, \widetilde{\lambda}_x^n - \chi_{hx}\big) + k\big(\omega_x^n + \lambda^n - (\omega_{hx}^n + \lambda_h^n), \widetilde{\lambda}^n - \chi_h\big) \\
&- \gamma\big(\vartheta^n - \vartheta_h^n, \widetilde{\lambda}_x^n - \chi_{hx}\big) + \big(\widetilde{\lambda}^n - \widetilde{\lambda}_h^n, \widetilde{\lambda}^n - \chi_h\big), \qquad (24)
\end{aligned}$$

$$\begin{aligned}
&\rho_3\left(\vartheta_t^n - \frac{\vartheta_h^n - \vartheta_h^{n-1}}{\Delta t}, \vartheta^n - \vartheta_h^n\right) + \big(q_x^n - q_{hx}^n, \vartheta^n - \vartheta_h^n\big) + \gamma\big(\widetilde{\lambda}_x^n - \widetilde{\lambda}_{hx}, \vartheta^n - \vartheta_h^n\big) \\
&= \rho_3\left(\vartheta_t^n - \frac{\vartheta_h^n - \vartheta_h^{n-1}}{\Delta t}, \vartheta^n - \eta_h\right) + \big(q_x^n - q_{hx}^n, \vartheta^n - \eta_h\big) + \gamma\big(\widetilde{\lambda}_x^n - \widetilde{\lambda}_{hx}, \vartheta^n - \eta_h\big), \qquad (25)
\end{aligned}$$

$$\tau\left(q_t^n - \frac{q_h^n - q_h^{n-1}}{\triangle t}, q^n - q_h^n\right) + \beta\big(q^n - q_h^n, q^n - q_h^n\big) + \big(\vartheta_x^n - \vartheta_{hx}^n, q^n - q_h^n\big)$$

$$= \tau\left(q_t^n - \frac{q_h^n - q_h^{n-1}}{\triangle t}, q^n - \alpha_h\right) + \beta\big(q^n - q_h^n, q^n - \alpha_h\big) + \big(\vartheta_x^n - \vartheta_{hx}^n, q^n - \alpha_h\big). \quad (26)$$

Step 2: Using that (13), the first term in equation (23) becomes

$$\left(\widetilde{\omega}_t^n - \frac{\widetilde{\omega}_h^n - \widetilde{\omega}_h^{n-1}}{\Delta t}, \widetilde{\omega}^n - \widetilde{\omega}_h^n\right) = \left(\widetilde{\omega}_t^n - \frac{\widetilde{\omega}^n - \widetilde{\omega}^{n-1}}{\Delta t}, \widetilde{\omega}^n - \widetilde{\omega}_h^n\right) + \frac{\left(\widetilde{\omega}^n - \widetilde{\omega}^{n-1} - \left(\widetilde{\omega}_h^n - \widetilde{\omega}_h^{n-1}\right), \widetilde{\omega}^n - \widetilde{\omega}_h^n\right)}{\Delta t}$$
$$= \left(\widetilde{\omega}_t^n - \frac{\widetilde{\omega}^n - \widetilde{\omega}^{n-1}}{\Delta t}, \widetilde{\omega}^n - \widetilde{\omega}_h^n\right) + \frac{1}{2\Delta t}\left\|\widetilde{\omega}^n - \widetilde{\omega}_h^n - \left(\widetilde{\omega}^{n-1} - \widetilde{\omega}_h^{n-1}\right)\right\|^2$$
$$+ \frac{1}{2\Delta t}\left(\left\|\widetilde{\omega}^n - \widetilde{\omega}_h^n\right\|^2 - \left\|\widetilde{\omega}^{n-1} - \widetilde{\omega}_h^{n-1}\right\|^2\right). \quad (27)$$

Then

$$\left(\widetilde{\omega}_t^n - \frac{\widetilde{\omega}_h^n - \widetilde{\omega}_h^{n-1}}{\Delta t}, \widetilde{\omega}^n - \widetilde{\omega}_h^n\right) \geqslant \left(\widetilde{\omega}_t^n - \frac{\widetilde{\omega}^n - \widetilde{\omega}^{n-1}}{\Delta t}, \widetilde{\omega}^n - \widetilde{\omega}_h^n\right) + \frac{\left(\left\|\widetilde{\omega}^n - \widetilde{\omega}_h^n\right\|^2 - \left\|\widetilde{\omega}^{n-1} - \widetilde{\omega}_h^{n-1}\right\|^2\right)}{2\Delta t}. \quad (28)$$

In the same way, for (24)–(26) we find

$$\left(\widetilde{\lambda}_t^n - \frac{\widetilde{\lambda}_h^n - \widetilde{\lambda}_h^{n-1}}{\Delta t}, \widetilde{\lambda}^n - \widetilde{\lambda}_h^n\right) \geqslant \left(\widetilde{\lambda}_t^n - \frac{\widetilde{\lambda}^n - \widetilde{\lambda}^{n-1}}{\Delta t}, \widetilde{\lambda}^n - \widetilde{\lambda}_h^n\right) + \frac{\left(\left\|\widetilde{\lambda}^n - \widetilde{\lambda}_h^n\right\|^2 - \left\|\widetilde{\lambda}^{n-1} - \widetilde{\lambda}_h^{n-1}\right\|^2\right)}{2\Delta t}, \quad (29)$$

$$\left(\vartheta_t^n - \frac{\vartheta_h^n - \vartheta_h^{n-1}}{\Delta t}, \vartheta^n - \vartheta_h^n\right) \geqslant \left(\vartheta_t^n - \frac{\vartheta^n - \vartheta^{n-1}}{\Delta t}, \vartheta^n - \vartheta_h^n\right) + \frac{\left(\|\vartheta^n - \vartheta_h^n\|^2 - \|\vartheta^{n-1} - \vartheta_h^{n-1}\|^2\right)}{2\Delta t}, \quad (30)$$

$$\left(q_t^n - \frac{q_h^n - q_h^{n-1}}{\Delta t}, q^n - q_h^n\right) \geqslant \left(q_t^n - \frac{q^n - q^{n-1}}{\Delta t}, q^n - q_h^n\right) + \frac{\left(\|q^n - q_h^n\|^2 - \|q^{n-1} - q_h^{n-1}\|^2\right)}{2\Delta t}, \quad (31)$$

using again (18) for $u^n = \lambda_x^n - \lambda_{hx}^n$, $\omega_x^n + \lambda^n - (\omega_{hx}^n + \lambda_h^n)$ and adding (23)–(26) we obtain

$$\frac{\rho_1}{2\Delta t}\left(\left\|\widetilde{\omega}^n - \widetilde{\omega}_h^n\right\|^2 - \left\|\widetilde{\omega}^{n-1} - \widetilde{\omega}_h^{n-1}\right\|^2\right) + \frac{\rho_2}{2\Delta t}\left(\left\|\widetilde{\lambda}^n - \widetilde{\lambda}_h^n\right\|^2 - \left\|\widetilde{\lambda}^{n-1} - \widetilde{\lambda}_h^{n-1}\right\|^2\right)$$
$$+ \frac{b}{2\Delta t}\left(\left\|\lambda_x^n - \lambda_{hx}^n\right\|^2 - \left\|\lambda_x^{n-1} - \lambda_{hx}^{n-1}\right\|^2\right)$$
$$+ \frac{k}{2\Delta t}\left(\|\omega_x^n + \lambda^n - (\omega_{hx}^n + \lambda_h^n)\|^2 - \left\|\omega_x^{n-1} + \lambda^{n-1} - \left(\omega_{hx}^{n-1} + \lambda_h^{n-1}\right)\right\|^2\right)$$
$$+ \frac{\rho_3}{2\Delta t}\left(\left\|\vartheta^n - \vartheta_h^n\right\|^2 - \left\|\vartheta^{n-1} - \vartheta_h^{n-1}\right\|^2\right) + \frac{\tau}{2\Delta t}\left(\left\|q^n - q_h^n\right\|^2 - \left\|q^{n-1} - q_h^{n-1}\right\|^2\right)$$
$$\leqslant C\Bigg(\left\|\widetilde{\omega}_t^n - \frac{\widetilde{\omega}^n - \widetilde{\omega}^{n-1}}{\Delta t}\right\|^2 + \frac{\left(\widetilde{\omega}^n - \widetilde{\omega}^{n-1} - \left(\widetilde{\omega}_h^n - \widetilde{\omega}_h^{n-1}\right), \widetilde{\omega}^n - \zeta_h\right)}{\Delta t} + \left\|\widetilde{\omega}^n - \widetilde{\omega}_h^n\right\|^2 + \left\|\widetilde{\omega}^n - \zeta_h\right\|^2$$
$$+ \left\|\widetilde{\omega}_x^n - \zeta_{hx}\right\|^2 + \left\|\omega_x^n + \lambda^n - (\omega_{hx}^n + \lambda_h^n)\right\|^2 + \left\|\widetilde{\lambda}_t^n - \frac{\widetilde{\lambda}^n - \widetilde{\lambda}^{n-1}}{\Delta t}\right\|^2 + \left\|\widetilde{\lambda}^n - \chi_h\right\|^2 + \left\|\widetilde{\lambda}^n - \widetilde{\lambda}_h^n\right\|^2$$
$$+ \left\|\lambda_x^n - \chi_{hx}\right\|^2 + \left\|\lambda_x^n - \lambda_{hx}^n\right\|^2 + \frac{\left(\widetilde{\lambda}^n - \widetilde{\lambda}^{n-1} - \left(\widetilde{\lambda}_h^n - \widetilde{\lambda}_h^{n-1}\right), \widetilde{\lambda}^n - \chi_h\right)}{\Delta t} + \left\|\vartheta_t - \frac{\vartheta^n - \vartheta^{n-1}}{\Delta t}\right\|^2$$
$$+ \left\|\vartheta^n - \vartheta_h^n\right\|^2 + \left\|\vartheta_x^n - \eta_x^n\right\|^2 + \frac{\left(\vartheta^n - \vartheta^{n-1} - (\vartheta_h^n - \vartheta_h^{n-1}), \vartheta^n - \eta_h\right)}{\Delta t} + \left\|\vartheta^n - \eta_h^n\right\|^2 + \left\|\vartheta_x^n - \eta_{hx}^n\right\|^2$$
$$+ \left\|q_t - \frac{q^n - q^{n-1}}{\Delta t}\right\|^2 + \frac{\left(q^n - q^{n-1} - (q_h^n - q_h^{n-1}), q^n - \alpha_h\right)}{\Delta t} + \|q^n - q_h^n\|^2 + \|q^n - \alpha_h^n\|^2 + \|q_x^n - \alpha_{hx}^n\|^2 \quad (32)$$

Step 3. By summing over n and multiplying the last inequality by Δt we have, $\forall\{\zeta_h, \chi_h, \eta_h, \alpha_h\}_{i=0}^N \in S^h$,

$$\|\widetilde{\omega}^n-\widetilde{\omega}_h^n\|^2+\|\widetilde{\lambda}^n-\widetilde{\lambda}_h^n\|^2+\|\widetilde{\lambda}_x^n-\widetilde{\lambda}_{hx}^n\|^2+\|\omega_x^n+\lambda^n-(\omega_{hx}^n+\lambda_h^n)\|^2+\|\vartheta^n-\vartheta_h^n\|^2+\|q^n-q_h^n\|^2$$
$$\leqslant C\Delta t\sum_{i=1}^{N}\Bigg(\|\widetilde{\omega}^i-\widetilde{\omega}_h^i\|^2+\|\widetilde{\lambda}^i-\widetilde{\lambda}_h^i\|^2+\|\widetilde{\lambda}_x^i-\widetilde{\lambda}_{hx}^i\|^2+\|\omega_{hx}^i+\lambda_h^i-(\omega_x^i+\lambda^i)\|^2+\|\vartheta^i-\vartheta_h^i\|^2$$
$$+\|q^i-q_h^i\|^2+\left\|\widetilde{\omega}_t^i-\frac{\widetilde{\omega}^i-\widetilde{\omega}^{i-1}}{\Delta t}\right\|^2+\frac{\left(\widetilde{\omega}^i-\widetilde{\omega}^{i-1}-(\widetilde{\omega}_h^i-\widetilde{\omega}_h^{i-1}),\widetilde{\omega}^i-\zeta_h^i\right)}{\Delta t}+\left\|\widetilde{\lambda}_t^i-\frac{\widetilde{\lambda}^i-\widetilde{\lambda}^{i-1}}{\Delta t}\right\|^2$$
$$+\frac{\left(\widetilde{\lambda}^i-\widetilde{\lambda}^{i-1}-(\widetilde{\lambda}_h^i-\widetilde{\lambda}_h^{i-1}),\widetilde{\lambda}^i-\chi_h^i\right)}{\Delta t}+\left\|\vartheta_t^i-\frac{\vartheta^i-\vartheta^{i-1}}{\Delta t}\right\|^2+\left\|q_t^i-\frac{q^i-q^{i-1}}{\Delta t}\right\|^2$$
$$+\frac{\left(\vartheta^i-\vartheta^{i-1}-(\vartheta_h^i-\vartheta_h^{i-1}),\vartheta^i-\eta_h^i\right)}{\Delta t}+\frac{\left(q^i-q^{i-1}-(q_h^i-q_h^{i-1}),q^i-\alpha_h^i\right)}{\Delta t}+\|\widetilde{\omega}^i-\zeta_h^i\|^2+\|\widetilde{\omega}_x^i-\zeta_{hx}^i\|^2$$
$$+\|\widetilde{\lambda}^i-\chi_h^i\|^2+\|\widetilde{\lambda}_x^i-\chi_{hx}^i\|^2+\|\vartheta_x^i-\eta_{hx}^i\|^2+\|q_x^i-\alpha_{hx}^i\|^2+\|\vartheta^i-\eta_h^i\|^2+\|q^i-\alpha_h^i\|^2\Bigg)$$
$$+C\Big(\|\omega^1-\widetilde{\omega}_h^0\|^2+\|\lambda^1-\widetilde{\lambda}_h^0\|^2+\|\lambda_x^0-\lambda_{hx}^0\|^2+\|\omega_{hx}^0+\lambda_h^0-(\omega_x^0+\lambda^0)\|^2+\|\vartheta^0-\vartheta_n^0\|^2+\|q^0-q_n^0\|^2\Big).$$

Considering that (with an analogous outcome for comparable expressions) [12]

$$\sum_{i=1}^{N}\left(\widetilde{\omega}^i-\widetilde{\omega}^{i-1}-\left(\widetilde{\omega}_h^i-\widetilde{\omega}_h^{i-1}\right),\widetilde{\omega}^i-\zeta_h^i\right)=\left(\widetilde{\omega}^N-\widetilde{\omega}_h^N,\widetilde{\omega}^N-\zeta_h^N\right)+\left(\widetilde{\omega}_h^0-\widetilde{\omega}^0,\widetilde{\omega}^1-\zeta_h^1\right)$$
$$+\sum_{i=1}^{N-1}\left\|\widetilde{\omega}_h^i-\widetilde{\omega}_h^i,\widetilde{\omega}^i-\zeta_h^i-\left(\widetilde{\omega}^{i+1}-\zeta_h^{i+1}\right)\right\|\leqslant C\left(\|\widetilde{\omega}^N-\widetilde{\omega}_h^N\|^2+\|\widetilde{\omega}^N-\zeta_h^N\|^2+\|\widetilde{\omega}_h^0-\zeta_h^0\|^2+\|\widetilde{\omega}^1-\zeta_h^1\|^2\right)$$
$$+C\Delta t\sum_{i=1}^{N-1}\|\widetilde{\omega}^i-\widetilde{\omega}_h^i\|^2+\frac{C}{\Delta t}\sum_{i=1}^{N-1}\left\|\widetilde{\omega}^i-\zeta_h^i-\left(\widetilde{\omega}^{i+1}-\zeta_h^{i+1}\right)\right\|^2,\quad(33)$$

as well as utilizing a discrete form of Gronwall's inequality [13], the result follows. ■

The numerical method's convergence is stated in the following consequence.

Corollary 1. *If we assume that the continuous problem's solution is sufficiently regular, that means:*

$$\omega,\lambda\in H^3\left(0,T;L^2(0,L)\right)\cap W^{1,\infty}\left(0,T;H^1(0,L)\right)\cap H^2\left(0,T;H^1(0,L)\right),\quad(34)$$

and,

$$\vartheta,q\in H^2\left(0,T;L^2(0,L)\right)\cap L^\infty\left(0,T;H^2(0,L)\right)\cap H^1\left(0,T;H^1(0,L)\right).\quad(35)$$

Then, there exists a positive constant C, independent of the discretization parameters h and Δt, such that:

$$\|\widetilde{\omega}^n-\widetilde{\omega}_h^n\|^2+\|\widetilde{\lambda}^n-\widetilde{\lambda}_h^n\|^2+\|\widetilde{\lambda}_x^n-\widetilde{\lambda}_{hx}^n\|^2+\|\omega_x^n+\lambda^n-(\omega_{hx}^n+\lambda_h^n)\|^2+\|\vartheta^n-\vartheta_h^n\|^2+\|q^n-q_h^n\|^2$$
$$\leqslant C\left(h^2+\Delta t^2\right).\quad(36)$$

Proof. The result is a consequence of following estimates as in [14] and [15]:
$\frac{1}{\Delta t}\sum_{n=1}^{N-1}\left\|\widetilde{\omega}^n-\zeta_h^n-\left(\widetilde{\omega}^{n+1}-\zeta_h^{n+1}\right)\right\|\leqslant Ch^2\|\widetilde{\omega}_t\|^2_{L^2(0,T;H^1(0,L))}$. ■

2.3. Numerical simulation

We implement several numerical experiments in this part to demonstrate the theoretical findings in Theorems 1 and Corollary 1. To accomplish this, we discretize Timoshenko system with the second sound using a finite element method (FEM) for space and using the backward Euler scheme for time for domain $[0,1]\times[0,T]$ to find solution of (37) and (38).

Assuming that $\widetilde{\omega}_h^{n-1}$, $\widetilde{\lambda}_h^{n-1}$, ϑ_h^{n-1}, q_h^{n-1} are known and let $\omega_h^{n,0}=\omega_h^{n-1}$, $\widetilde{\omega}_h^{n,0}=\widetilde{\omega}_h^{n-1}$, $\lambda_h^{n,0}=\lambda_h^{n-1}$, $\widetilde{\lambda}_h^{n,0}=\widetilde{\lambda}_h^{n-1}$, $\vartheta_h^{n,0}=\vartheta_h^{n-1}$, $q_h^{n,0}=q_h^{n-1}$, we will solve the following system:

$$\frac{\rho_1}{\Delta t}\left(\widetilde{\omega}_h^{n,j}-\widetilde{\omega}_h^{n-1},\zeta_h\right)+k\left(\omega_{hx}^{n,j}+\lambda_h^{n,j-1},\zeta_{hx}\right)=0,$$

$$\frac{\rho_2}{\Delta t}\big(\widetilde{\lambda}_h^{n,j}-\widetilde{\lambda}_h^{n-1},\chi_h\big)+b\big(\lambda_{hx}^{n,j},\chi_{hx}\big)+k\big(\omega_{hx}^{n,j}+\lambda_h^{n,j},\chi_h\big)-\gamma\big(\vartheta_h^{n,j-1},\chi_{hx}\big)+\big(\widetilde{\lambda}_h^{n,j},\chi_h\big)=0,$$
$$\frac{\rho_3}{\Delta t}\big(\vartheta_h^{n,j}-\vartheta_h^{n-1},\eta_h\big)+\big(q_{hx}^{n,j-1},\eta_h\big)+\gamma\big(\widetilde{\lambda}_x^{n,j},\eta_h\big)=0, \tag{37}$$
$$\frac{\tau}{\Delta t}\big(q_h^{n,j}-q_h^{n-1},\alpha_h\big)+\beta\big(q_h^{n,j},\alpha_h\big)+\big(\vartheta_{hx}^{n,j},\alpha_h\big)=0,$$

where, for $j=1,2,\ldots,J$

$$\omega_h^{n,j}=\omega_h^{n-1}+\Delta t\widetilde{\omega}_h^{n,j},\quad \lambda_h^{n,j}=\lambda_h^{n-1}+\Delta t\widetilde{\lambda}_h^{n,j}. \tag{38}$$

The time interval $(0,T)$ is splitted into N subintervals with a time step $\Delta t=\frac{T}{N}$, whereas the spatial interval $(0,1)$ is divided into J subintervals.

We show the energy decay for both cases $\mu\neq 0$ and $\mu=0$ presented in Example 1 and Example 2 and we perform the numerical convergence in Example 3.

For three numerical evaluations, we take the functions σ and h defined by: $\sigma(t)=1$ and $f(z)=z$.

Example 1 (The case $\mu\neq 0$). We consider first the following parameters of the model

$$\rho_1=2,\quad \rho_2=2,\quad \rho_3=1,\quad b=2,\quad k=2,\quad \tau=1,\quad \gamma=1,\quad \beta=1. \tag{39}$$

The discretization parameters are

$$\Delta t=\frac{1}{50},\quad h=\frac{1}{J},\quad J=200,\quad T=6. \tag{40}$$

Along with the following initial conditions

$$\begin{cases}\omega_0(x)=\sin(\frac{\pi}{2}x)x^2(1-x)^2=\omega_t(x,0)=\omega_1(x), & \forall x\in(0,1),\\ \lambda_0(x)=\cos(\pi x)x^3(1-x)^3=\lambda_t(x,0)=\lambda_1(x), & \forall x\in(0,1),\\ \vartheta_0(x)=\vartheta(x,0)=\exp(-20x)\,x^4(1-x)^4, & \forall x\in(0,1),\\ q_0(x)=q(x,0)=x^4(1-x)^4, & \forall x\in(0,1).\end{cases} \tag{41}$$

We observe that a polynomial decay of the energy is showed in Figures 1, 2 for the case of $\mu\neq 0$ that goes with theoretical results.

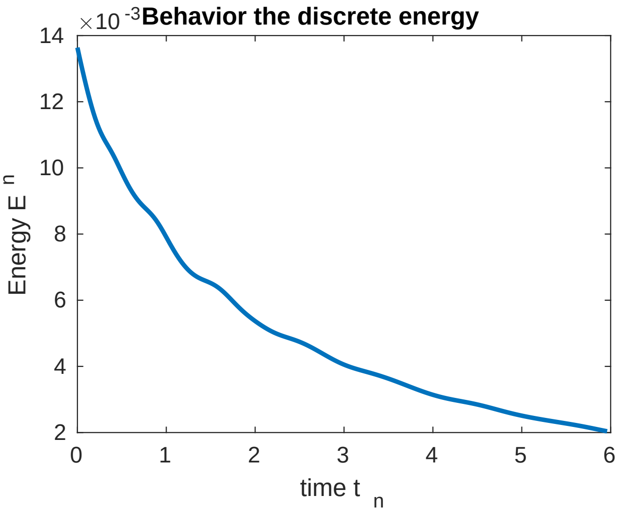

Fig. 1. Numerical energy of the system in the case of $\mu\neq 0$.

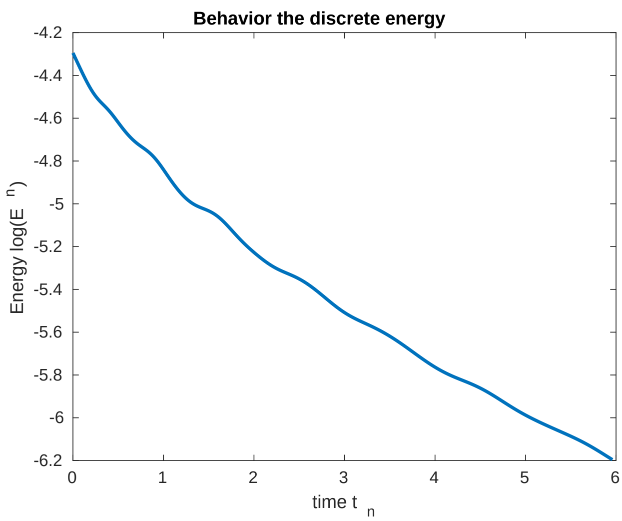

Fig. 2. $\ln(E^n)$ in the case of $\mu\neq 0$.

Example 2 (The case $\mu=0$). We consider the following parameters of the model

$$\rho_1=10^{-2},\quad \rho_2=10^{-2},\quad \rho_3=10^{-2},\quad k=0.2,\quad b=1,\quad \beta=1,\quad \gamma=\sqrt{3.992},\quad \tau=10^{-2} \tag{42}$$

with the same discretization parameters (40) and initial conditions (41).

We found an exponential decay of the Timoshenko system's energy in Figure 3, 4 for the case of $\mu=0$ then the theoretical results are verified.

From the numerical solution in three dimensions of the diplacement ω and the filament λ obtained from Figure 5, 6. This proves again the energy decay of the system.

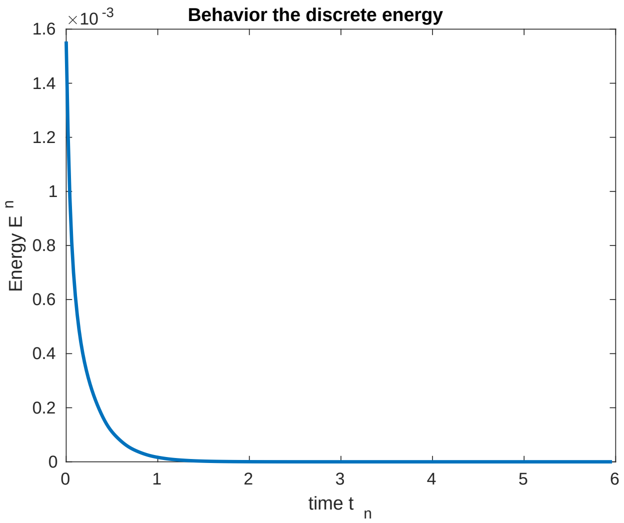

Fig. 3. Numerical energy of the system in the case of $\mu = 0$.

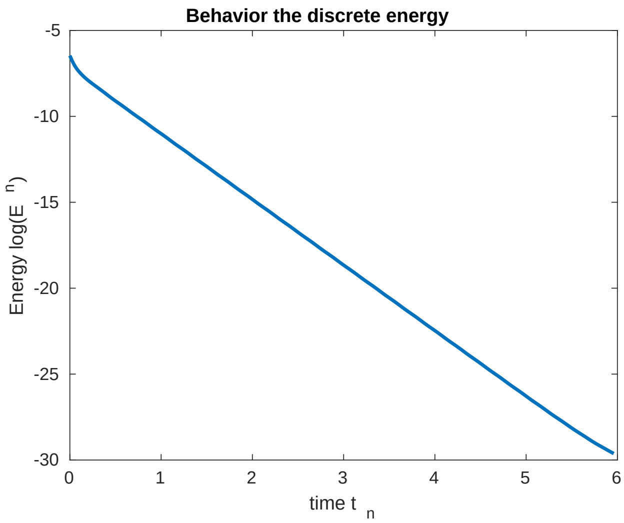

Fig. 4. $\ln(E^n)$ in the case of $\mu = 0$.

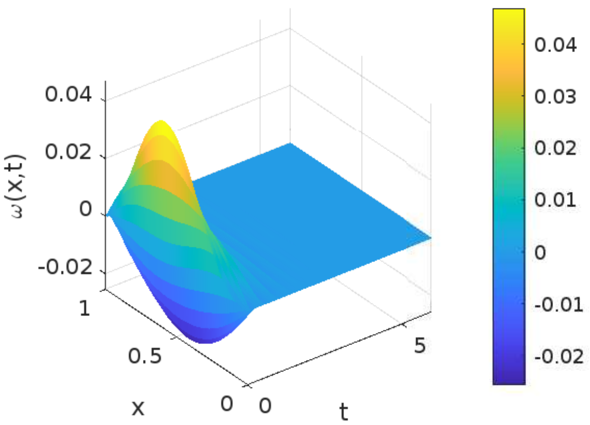

Fig. 5. The evolution in time and space of ω.

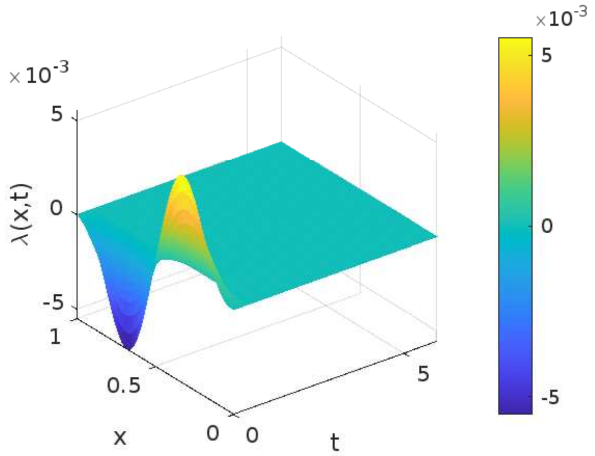

Fig. 6. The evolution in time and space of λ.

In Figure 7 the rotation angle of filament λ is shown at several time instants. Moreover, in Figure 8 the evolution in time of the rotation angle of filament λ at several space points. As expected, the rotation angle λ is generated initially but it converges to zero.

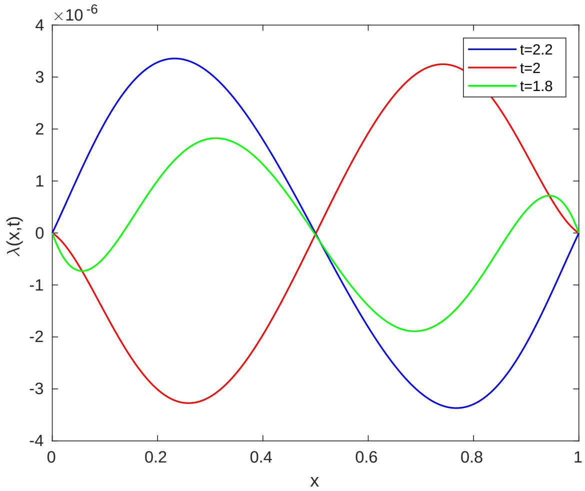

Fig. 7. The evolution in space of λ for $t = 1.8$, $t = 2$ and $t = 2.2$.

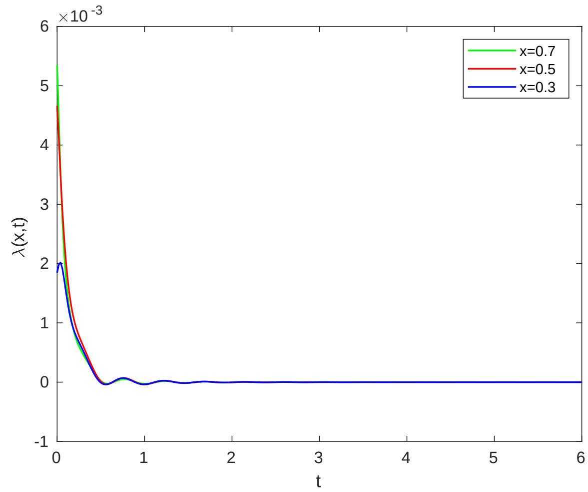

Fig. 8. The evolution in time of λ for $x = 0.3$, $x = 0.5$ and $x = 0.7$.

Example 3 (Order of convergence). The main objective of this example is to demonstrate the numerical convergence of the numerical scheme. The next system is addressed in this context. Following that, we perform a simulation to numerically test the error estimate. We have the modified problem (P).

Find, $\omega\colon [0,1]\times[0,T]\to\mathbb{R}$, $\lambda\colon [0,1]\times[0,T]\to\mathbb{R}$, $\vartheta\colon [0,1]\times[0,T]\to\mathbb{R}$ and $q\colon [0,1]\times[0,T]\to\mathbb{R}$ such that

$$(P)\quad\begin{cases}\rho_1\omega_{tt}-k(\omega_x+\lambda)_x=f_1 & \text{in } (0,1)\times(0,T),\\ \rho_2\lambda_{tt}-b\lambda_{xx}+k(\omega_x+\lambda)+\gamma\vartheta_x+\lambda_t=f_2 & \text{in } (0,1)\times(0,T),\\ \rho_3\vartheta_t+q_x+\gamma\lambda_{xt}=f_3 & \text{in } (0,1)\times(0,T),\\ \tau q_t+\beta q+\vartheta_x=f_4 & \text{in } (0,1)\times(0,T),\end{cases}$$

where f_1, f_2, f_3, f_4, and the initial data are derived from the exact solution:
for all $(x,t)\in(0,1)\times(0,T)$

$$\omega(x,t)=e^tx^2(x-1)^2,\qquad \lambda(x,t)=e^tx^2(x-1)^2,$$
$$\vartheta(x,t)=e^tx^2(x-1)^2,\qquad q(x,t)=e^tx^2(x-1)^2.$$

For computed errors we have adopted the values of parameters in (43),

$$\rho_1=1,\quad \rho_2=10,\quad \rho_3=10^3,\quad k=10^{-2},\quad b=10^4,\quad \beta=1.3\cdot10^6,\quad \gamma=2\cdot10^{-4},\quad \tau=10^4. \tag{43}$$

Table 1 displays the computed errors at time $T=1$, where the Error is supplied by

$$\text{Error}=\max_{0\leqslant n\leqslant N}\left\{\left\|\widetilde{\omega}^n-\widetilde{\omega}_h^n\right\|^2+\left\|\widetilde{\lambda}^n-\widetilde{\lambda}_h^n\right\|^2+\left\|\widetilde{\lambda}_x^n-\widetilde{\lambda}_{hx}^n\right\|^2+\left\|\omega_x^n+\lambda^n-\left(\omega_{hx}^n+\lambda_h^n\right)\right\|^2+\left\|\vartheta^n-\vartheta_h^n\right\|^2+\left\|q^n-q_h^n\right\|^2\right\}.$$

Table 1. Errors computed for $T=1$.

J	Δt	Error
20	2×10^{-2}	2.3410×10^{-3}
40	10^{-2}	70305×10^{-4}
80	5×10^{-3}	2.1621×10^{-4}
160	2.5×10^{-3}	7.4839×10^{-5}
320	1.25×10^{-3}	3.2442×10^{-5}

The errors generated for different discretization parameters J (with $h=\frac{1}{J}$) and Δt in Table 1. We can also see the numerical convergence in Corollary 1 is achieved according to Figures 9, 10.

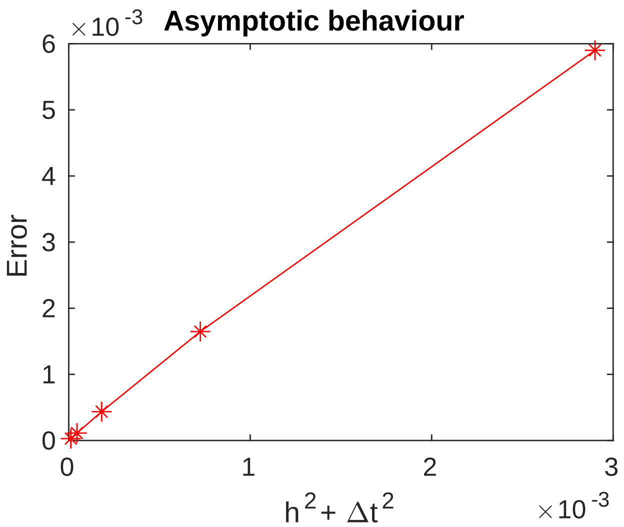

Fig. 9. The evolution of Error.

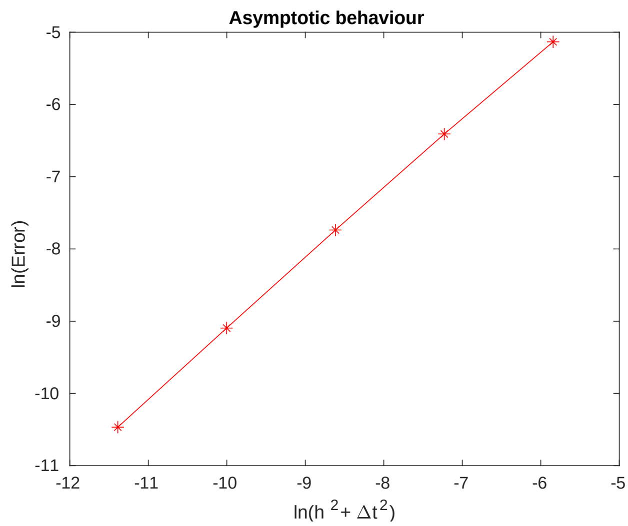

Fig. 10. The evolution of ln(Error).

The numerical schemes were implemented using MATLAB on a Intel Core i5-6006U CPU @ 2.00 GHz.

3. Conclusion

The numerical analysis of the Timoshenko system with the second sound was done in this paper. In order to approach discrete energy, we first devised a numerical scheme based on finite element discretization in the space variable and the finite difference scheme in time. Additionally, we showed the energy decay propriety. The semi-discrete and completely discrete schemes a priori error estimates are then proved. At the end, several numerical tests were performed for this system, and the results show that the convergence matches what is predicted by the theories.

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Чисельні дослідження системи Тимошенка з другим звуком

Смук А., Радід А.

Факультет математики та інформатики Університету Хасана II,
FSAC, Лабораторія фундаментальної та прикладної математики, Касабланка, Марокко

Задача Тимошенко не є новою задачею, існує багато статей, які присвячені її дослідженню. З'являються нові фізичні задачі, які вимагають хорошого математичного розуміння поведінки цього явища. Наш внесок полягатиме у вивченні чисельної стійкості системи Тимошенка з другим звуком. Вводимо наближення скінченних елементів і доводимо, що відповідна дискретна енергія зменшується, і встановлюємо апріорні оцінки похибок. Накінець, отримуємо декілька чисельних симуляцій.

Ключові слова: *задача Тимошенка; чисельна стійкість; метод скінченних елементів; чисельне моделювання.*