MATHEMATICAL MODELING AND COMPUTING, Vol. 11, No. 4, pp. 923–929 (2024)

Blind image deblurring using Nash game and the fractional order derivative

Semmane F. Z.[1], Moussaid N.[1], Ziani M.[2]

[1] *LMCSA laboratory, Hassan II University of Casablanca, P.O. Box 146, Mohammedia, Morocco*
[2] *LMSA laboratory, Department of Mathematics, Faculty of Sciences, Mohammed V University in Rabat, Morocco*

(Received 8 December 2023; Revised 19 July 2024; Accepted 22 July 2024)

This paper presents an innovative approach to blind image deblurring based on fractional order derivatives and Nash game theory. The integration of fractional order derivatives enhances the deblurring process, capturing intricate image details beyond the capabilities of traditional integer-order derivatives. The Nash game framework is employed to model the strategic interaction between the image and the unknown blur kernel, fostering a cooperative optimization process. Experimental results showcase the proposed method's superiority in terms of both Peak Signal-to-Noise Ratio (PSNR) and Structural Similarity Index (SSIM) when compared to existing methods. The fractional order derivative enhances image structure preservation, while the Nash game facilitates joint optimization of image restoration and blur kernel estimation.

Keywords: *deconvolution; image deblurring; optimisation; Nash game; total variation regularization; fractional order derivative.*

2010 MSC: 90C29, 91A05, 58E17, 65K10 **DOI:** 10.23939/mmc2024.04.923

1. Introduction

Blind deconvolution is a challenging problem in the field of image processing, with applications in various domains such as astronomy, microscopy, medical imaging, and forensics [1]. This problem arises when we have an observed image that is a result of convolving an unknown image (the source or object) with an unknown point spread function (PSF) or kernel, and our goal is to recover both the original image and the PSF from the blurred observation. Blind deconvolution can be formulated mathematically as follows, see [2]:

$$i = k * v + n.$$

Here: i is blurred and noisy observed image, v is unknown true image, k is unknown blurring kernel, n is additive noise.

Several methods have been developed to solve this problem as the method proposed by Rudin et al. [3], where the image recovery problem is stated as follows

$$\min_v J(v) = \frac{1}{2}\|k * v - i\|^2_{L^2(\gamma)} + \int_\gamma |\nabla v| \, dx \, dy.$$

Another approach suggested for addressing the blind deconvolution (BD) problem, proposed by Chan and Wong in their study [4], involves using the Total Variation (TV) norm, see also [5].

$$\min_{v,k} J(v,k) = \frac{1}{2}\|k * v - i\|^2_{L^2(\gamma)} + \alpha_1 \int_\gamma |\nabla v| \, dx \, dy + \alpha_2 \int_\gamma |\nabla k| \, dx \, dy.$$

They use the TV regularization, significantly, they employ Total Variation (TV) regularization on both the original image v and the blur kernel k with α_1 and α_2 are respective positive parameters. The total variation [5], can be defined as

$$\mathrm{TV}(f) = \sup\left\{\int_\gamma f \operatorname{div}\phi \colon \phi \in C_0^1 \text{ and } |\phi|_{L^\infty(\gamma)} \leqslant 1\right\}.$$

Meskine et al. [6] proposes a solution to the blind deconvolution problem using a game theory perspective. Specifically, they identify the Nash equilibrium to determine the optimal estimation of both the image and Point Spread Function (PSF), see [7]. In their approach, two functionals are involved and minimized

$$\begin{cases} J_v(v,k) = \dfrac{1}{2}\|k*v-i\|^2_{L^2(\gamma)} + \displaystyle\int_\gamma \alpha(x)|Dv|\,dx, \\ J_k(v,k) = \dfrac{1}{2}\|k*v-i\|^2_{L^2(\gamma)} + \displaystyle\int_\gamma (1-\alpha(x))|Dk|\,dx. \end{cases} \tag{1}$$

The blind deconvolution process becomes more complex when incorporating Nash game theory [8], and fractional order derivatives. Our goal in this paper is to examine the synergy between Nash game theory and fractional order derivatives in the context of blind deconvolution. By combining strategic modeling of the players involved with finer regularization [9], our approach aims to overcome the limitations of conventional methods and produce more accurate and robust deconvolution results. In the following sections, we will detail the theoretical foundations of Nash game theory and fractional order derivatives, exploring their respective application to blind deconvolution. We then present our research methodology, illustrate case studies and discuss numerical results.

2. Blind deconvolution using Nash game

Over the years, a variety of strategies and methods have been explored in the field of deblurring to enhance and recover images. These techniques are designed to address blurriness in images caused by factors such as motion, defocus, or other distortions [10]. In this paper, we propose to modify the mathematical model for blind image deblurring presented in [6], such that we use fractional order derivatives. Our problem to solve is

$$\begin{cases} J_v(v,k) = \dfrac{1}{2}\|k*v-i\|^2_{L^2(\gamma)} + \displaystyle\int_\gamma \alpha(x)|Dv|\,dx, \\ J_k(v,k) = \dfrac{1}{2}\|k*v-i\|^2_{L^2(\gamma)} + \displaystyle\int_\gamma (1-\alpha(x))|D^\beta k|\,dx, \end{cases} \tag{2}$$

with $\alpha(x)$ is a spatially and scale adapture function given by $\alpha(x) = \frac{1}{1+\lambda|\nabla G_\sigma * i|}$, where $G_\sigma(x) = \frac{1}{2\pi\sigma^2}\exp\left(-\frac{|x^2|}{2\sigma^2}\right)$ is the Gaussian filter with the parameter σ and D^β is the fractional order derivative and β could be a fractional number. The order of the fractional derivation can be chosen according to the expected characteristics of the image. A higher β favors greater regularity.

Calculating fractional order derivatives involves using specialized methods from fractional calculus. One common approach is to use the Grünwald–Letnikov derivative [11].

The β-order variation of $v\colon \gamma \to \mathbb{R}$ it enhances the conventional total variation (TV) approach by incorporating fractional order derivatives. Specifically, it relies on Grünwald–Letnikov fractional order derivatives to introduce a more nuanced and sophisticated dimension to the regularization process, the discrete fractional order gradient is defined as $\nabla^\beta v = \left[D_1^\beta v, D_2^\beta v\right]$, where

$$(D_1^\beta v)_{i,j} = \sum_{m=0}^{M+1} (-1)^m C_m^\beta v_{i-m,j} \quad \text{and} \quad (D_2^\beta v)_{i,j} = \sum_{m=0}^{M+1} (-1)^m C_m^\beta v_{i,j-m}.$$

And M is the number of neighboring pixels using to approximate the fractional order derivative at each pixel and the coefficient C_m^β is defined as

$$C_m^\beta = \frac{\Gamma(\beta+1)}{\Gamma(m+1)\,\Gamma(\beta-m+1)}.$$

$\Gamma(x)$ is function defined as $\Gamma(x) = \int_0^\infty t^{x-1}.e^{-t}$ for all $x > 0$.

Now, we can solve the problem (2) by dividing the optimization variables. We consider a game where the players act with diverse objectives. The first player chooses strategy v in order to minimize a function $J_v(v,k)$ and the second player chooses his strategy k in order to minimize a function $J_k(v,k)$.

Clearly, the objective functions depend on two domains. Therefore, the choice of strategies of one player influences the choices of the other one. Two players act concurrently until an equilibrium is found [12]. That means each player has minimized the function with a common pair of strategies:

$$\begin{cases} \text{Find } (v^*, k^*) \text{ such that} \\ \quad \min\limits_{v} J_v(v, k^*) = J_v(v^*, k^*), \\ \quad \min\limits_{k} J_k(v^*, k) = J_k(v^*, k^*). \end{cases} \tag{3}$$

Theorem 1. *There exists a Nash equilibrium (k^*, v^*) solution of the problem (3) [2].*

The iterative schemes for (3), can be found by using the corresponding first order optimality conditions as

$$\frac{\partial J_v}{\partial v} = \frac{\partial J_k}{\partial k} = 0. \tag{4}$$

To attain a physically meaningful solution, we must impose conditions on v and k. Consequently, we have opted to enforce the following conditions:

$$\int_{\gamma} k(x, y)\, dx\, dy = 1 \quad \text{and} \quad v(x, y),\ k(x, y) \geqslant 0.$$

The Nash equilibrium is calculated by the subsequent algorithm, as presented in [5].

Algorithm 1 Nash algorithm

1. Initialization: $m = 0$, the noisy image $v^{(0)}$ and the kernel $k^{(0)}$.
2. **Step 1:**
   Phase 1: Resolve the problem $\min\limits_{k} J_k(k, v^{(m)}) \longrightarrow k^{(m+1)}$.
   Phase 2: Find a solution to the problem $\min\limits_{v} J_v(k^{(m)}, v) \longrightarrow v^{(m+1)}$
3. **Step 2:**
   and repeat Step 1 until convergence.
   There is convergence when $\|k^{(m+1)} - k^{(m)}\| < \varepsilon$ and $\|v^{(m+1)} - v^{(m)}\| < \varepsilon$, where ε is to be specified.

3. Numerical results

We employed the MATLAB software to implement the proposed method. Initially, blurred and noisy images were synthesized by convolving with a 7×7 Gaussian kernel of $\sigma = 1$ and introducing Gaussian noise with a mean of 0 and a variance of 0.0001. The quality of restoration was evaluated using the peak signal-to-noise ratio (PSNR) [13] and the structural similarity index measure(SSIM) [5]. Series of deconvolution experiments were conducted, comparing the proposed method with Meskine et al.'s approach, the Weiner filter method, regularized filter method, and Lucy–Richardson [14]. We examine five images, illustrated in Figures 1–5, respectively, for the simulation. The quality of image restoration results is presented in Tables 1–5, corresponding to each representative image.

Table 1. Restoration of Cameraman image using the different methods.

Compare	Our method	Meskine method	Regularized filter	BD algorithm	Weiner Filter	LR algorithm
PSNR	26.3747	25.7391	20.0116	21.1876	25.0217	22.2625
SSIM	0.92652	0.85959	0.83744	0.7703	0.80001	0.75825

Table 2. Restoration of Barbara image using the different methods.

Compare	Our method	Meskine method	Regularized filter	BD algorithm	Weiner Filter	LR algorithm
PSNR	25.5564	24.4801	24.3658	22.3028	24.0879	24.5585
SSIM	0.85801	0.83562	0.81434	0.72601	0.81702	0.74318

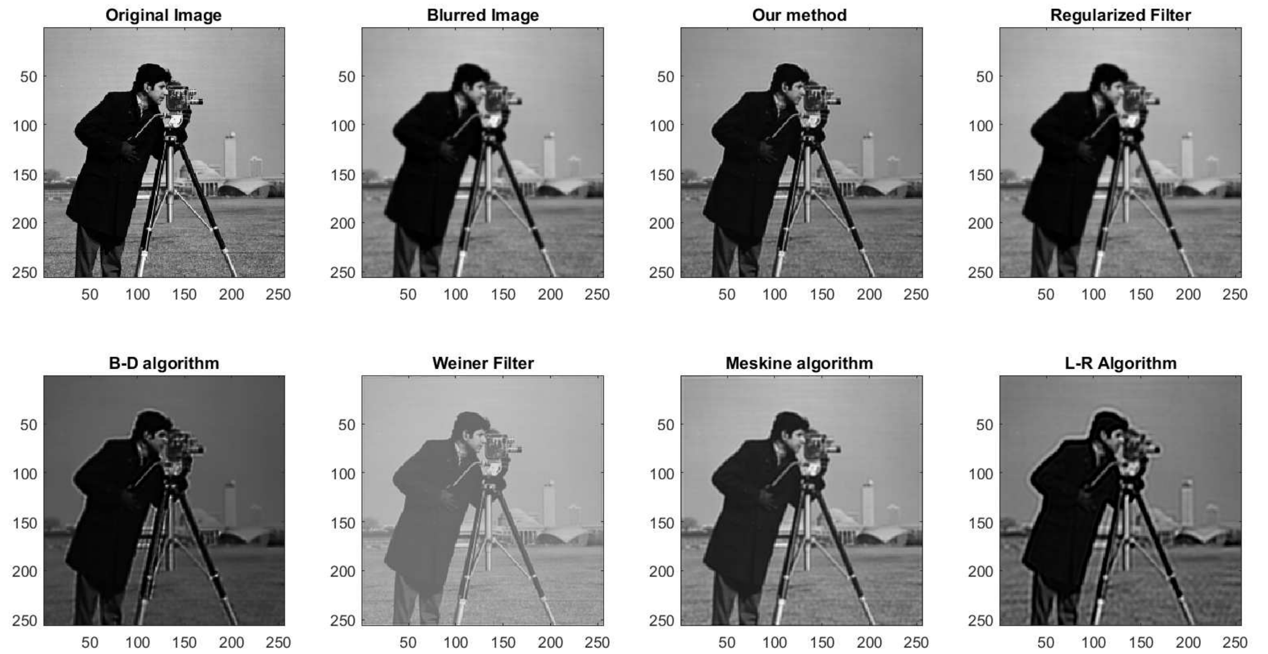

Fig. 1.

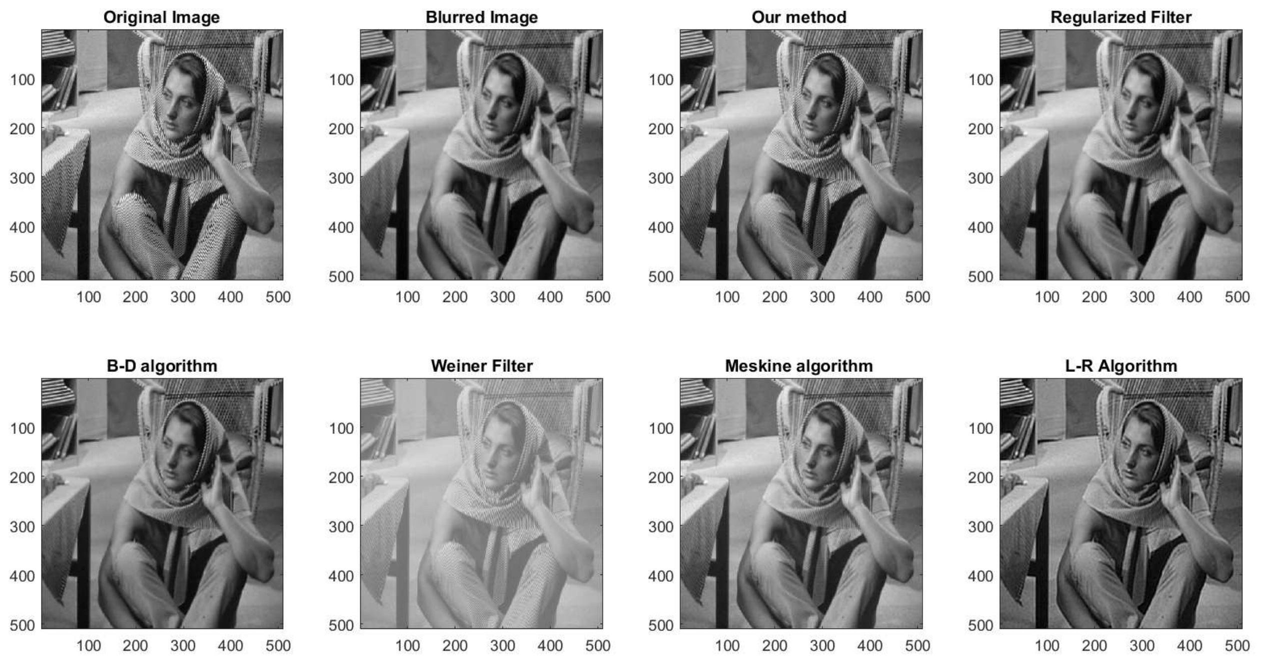

Fig. 2.

Table 3. Restoration of houses image using the different methods.

Compare	Our method	Meskine method	Regularized filter	BD algorithm	Weiner Filter	LR algorithm
PSNR	26.6139	24.8291	24.1342	22.2403	25.99320	25.3788
SSIM	0.8813	0.88008	0.82714	0.85453	0.80702	0.77279

Table 4. Restoration of peppers image using the different methods.

Compare	Our method	Meskine method	Regularized filter	BD algorithm	Weiner Filter	LR algorithm
PSNR	28.35270	26.069870	23.706542	27.45630	23.302851	27.20934
SSIM	0.87890	0.84324	0.82514	0.56241	0.88653	0.78510

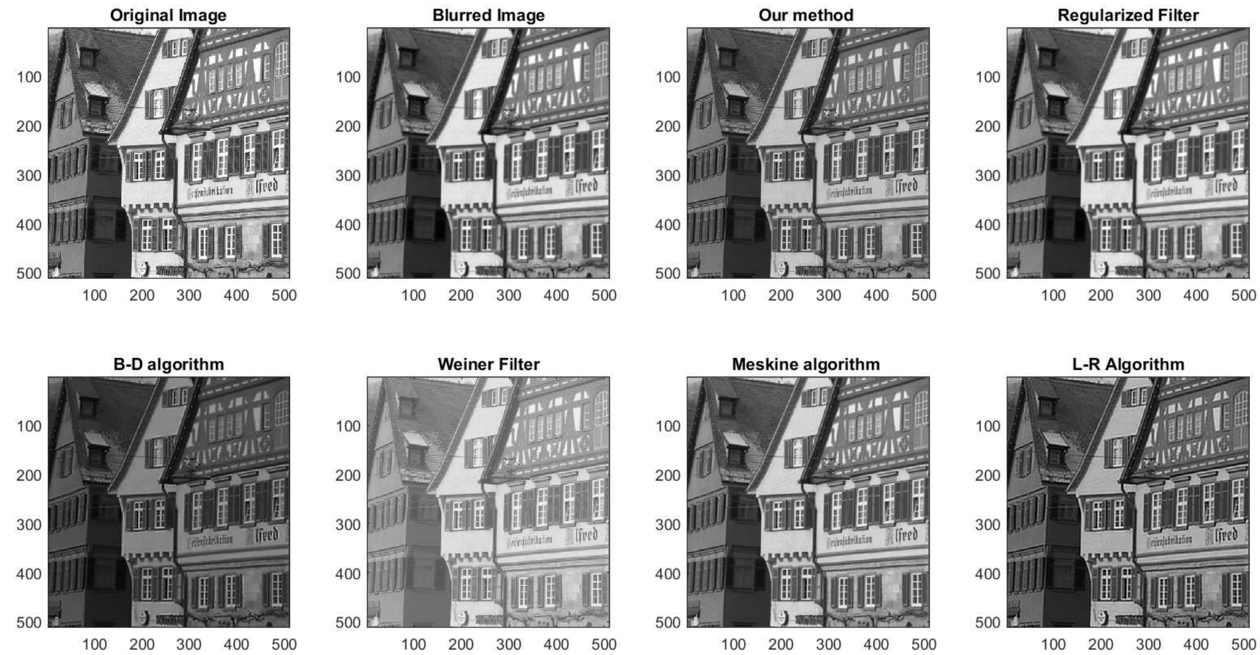

Fig. 3.

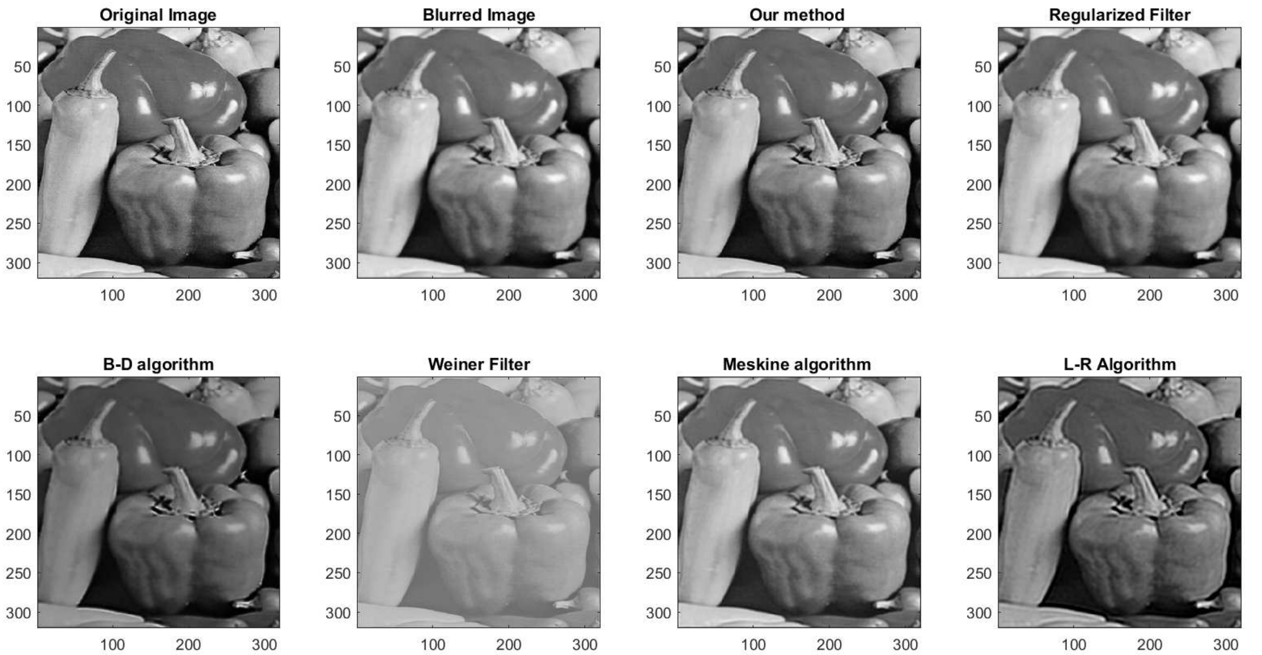

Fig. 4.

Table 5. Restoration of Lena image using the different methods.

Compare	Our method	Meskine method	Regularized filter	BD algorithm	Weiner Filter	LR algorithm
PSNR	27.8043	27.0532	22.5756513	26.16874	23.03421	22.73401
SSIM	0.87682	0.89563	0.71534	0.68421	0.84582	0.816034

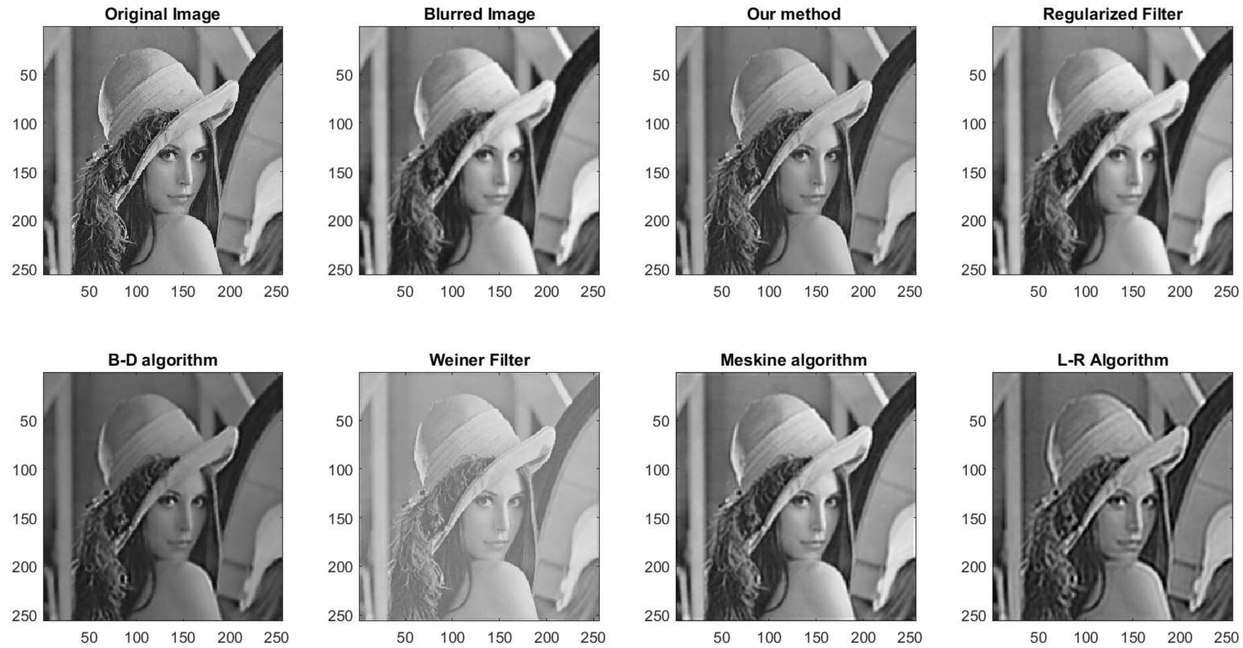

Fig. 5.

4. Conclusion

In this paper, we introduce a blind image deblurring method based on the Nash equilibrium and the fractional order derivative. We evaluate the proposed method against five different methods using grayscale images of the cameraman, Barbara, peppers, house, and Lena. The results show that the proposed method outperforms some methods in terms of PSNR and SSIM. The application of this approach to other areas such as 3D deconvolution, medical image restoration or satellite imagery could be explored. Adapting these techniques to more complex contexts would make it possible to test their robustness and generality. In summary, the use of the Nash game and the fractional derivative in blind image deconvolution opens up stimulating avenues of research. Continued exploration of these perspectives could not only enrich our understanding of deconvolution processes, but also lead to significant advances in the processing and analysis of complex images.

[1] Alaa H., Alaa N. E., Aqel F., Lefraich H. A new Lattice Boltzmann method for a Gray–Scott based model applied to image restoration and contrast enhancement. Mathematical Modeling and Computing. **9** (2), 187–202 (2022).

[2] Oldham K., Spanier J. The Fractional Calculus. Theory and Applications of Differentiation and Integration to Arbitrary Order. Academic Press (1974).

[3] Rudin L. I., Osher S., Fatemi E. Nonlinear total variation based noise removal algorithms. Physica D: Nonlinear Phenomena. **60** (1–4), 259–268 (1996).

[4] Chan T. F., Wong C.-K. Total variation blind deconvolution. IEEE Transactions on Image Processing. **7** (3), 370–375 (1998).

[5] Chan T., Esedoglu S., Park F., Yip A. Recent developments in total variation image restoration. Mathematical Models of Computer Vision. **17** (2), 17–31 (2005).

[6] Meskine D., Moussaid N., Berhich S. Blind image deblurring by game theory. NISS'19: Proceedings of the 2nd International Conference on Networking, Information Systems & Security. **31**, 1–7 (2019).

[7] Aboulaich R., Habbal A., Moussaid N. Optimisation multicritère: une approche par partage des variables. ARIMA. **13**, 77–89 (2010).

[8] Elmoumen S., Moussaid N., Aboulaich R. Image retrieval using Nash equilibrium and Kalai–Smorodinsky solution. Mathematical Modeling and Computing. **8** (4), 646–657 (2021).

[9] Semmane F. Z., Moussaid N., Ziani M. Searching for similar images using Nash game and machine learning. Mathematical Modeling and Computing. **11** (1), 239–249 (2024).

[10] Alaa K., Atounti M., Zirhem M. Image restoration and contrast enhancement based on a nonlinear reaction-diffusion mathematical model and divide and conquer technique. Mathematical Modeling and Computing. **8** (3), 549–559 (2021).

[11] Abdelouahab M.-S., Hamri N.-E. The Grünwald–Letnikov fractional-order derivative with fixed memory length. Mediterranean Journal of Mathematics. **13** (2), 557–572 (2016).

[12] Salah F.-E., Moussaid N. Machine learning and similar image-based techniques based on Nash game theory. Mathematical Modeling and Computing. **11** (1), 120–133 (2024).

[13] Wang Z., Bovik A. C., Sheikh H. R., Simoncelli E. P. Image quality assessment: from error visibility to structural similarity. IEEE Transactions on Image Processing. **13** (4), 600–612 (2004).

[14] Nasr N., Moussaid N., Gouasnouane O. The Kalai Smorodinsky solution for blind deconvolution. Computational and Applied Mathematics. **41** (5), 222 (2022).

Сліпе зменшення розмиття зображення за допомогою гри Неша та похідної дробового порядку

Семмане Ф. З.[1], Муссаїд Н.[1], Зіані М.[2]

[1]*Лабораторія LMCSA, Університет Хасана II Касабланки, п.с. 146, Мохаммедія, Марокко*
[2]*Лабораторія LMSA, кафедра математики факультету наук, Університет Мухаммеда V у Рабаті, Марокко*

У цій статті представлено інноваційний підхід до сліпого усунення розмитості зображень на основі дробових похідних і теорії ігор Неша. Інтеграція дробових похідних покращує процес усунення розмитості, фіксуючи складні деталі зображення, що перевершує можливості традиційних цілочисельних похідних. Ігровий фреймворк Неша використовується для моделювання стратегічної взаємодії між зображенням і невідомим ядром розмиття, сприяючи спільному процесу оптимізації. Експериментальні результати демонструють перевагу запропонованого методу як щодо пікового співвідношення сигнал–шум (PSNR), так і індексу структурної подібності (SSIM) порівняно з існуючими методами. Похідна дробового порядку покращує збереження структури зображення, тоді як гра Неша полегшує спільну оптимізацію відновлення зображення та оцінку ядра розмиття.

Ключові слова: *деконволюція; зменшення розмитості зображення; оптимізація; гра Неша; повна варіаційна регулярізація; похідна дробового порядку.*