MATHEMATICAL MODELING AND COMPUTING, Vol. 11, No. 4, pp. 954–965 (2024)

SOR Homotopy perturbation method to solve integro-differential equations

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(Received 28 December 2023; Accepted 25 May 2024)

We present in this paper, SOR Homotopy perturbation method, a new decomposition method by introducing a parameter ω to extend a classical homotopy perturbation method for solving integro-differential equations of various kinds. Using SOR homotopy perturbation method and its iterative scheme we can give the exact solution or a closed approximate to the solution of the problem. The convergence of the proposed method has been elaborated and some illustrative examples are presented with applications to Fredholm and Volterra integral equations.

Keywords: *SOR Homotopy perturbation method; Fredholm integral equation; Volterra integral equation.*

2010 MSC: 65R99, 40A10 **DOI:** 10.23939/mmc2024.04.954

1. Introduction

A variety of problems in physics, chemistry and biology have their mathematical setting as differential, integral or integro-differential equations [1]. In recent years, there has been a clear interest in integro-differential equations as a combination of differential and Volterra–Fredholm integral equations. Integro-differential equations play an important role in many branches of linear or nonlinear functional analysis and their applications. The mentioned integro-differential equations are usually difficult to solve analytically, so approximation strategies are required to obtain the solution of the linear and nonlinear integro-differential equations [2].

The Homotopy perturbation method (HPM) was first proposed by He J. Huan in 1999, where the solution of this method is considered as the sum of an infinite series which is very rapidly converge to the accurate solution [3]. The method deforms the difficult problem under study into a simple and easy to solve problem. He J. used the Homotopy perturbation method for solving nonlinear ordinary differential equations of the first and the second orders [3], nonlinear ordinary differential equations with n-th order [4], the oscillators equation with discontinuities [5], and one dimensional nonlinear wave equation [6]. In [7], the authors use the Homotopy perturbation method to solve the electrostatic potential differential equation.

There are many applications of Homotopy perturbation method to solve the reaction-diffusion equation [8], Gas dynamics equation [9], Schrodinger equation [10], delay differential equations [11], linear/nonlinear Volttera and Fredohlm equations [12, 13].

Our contribution here can be summarized in the following points:

- introducing a parameter ω to define a SOR Homotopy perturbation method;
- the convergence of the proposed method is discussed;
- some Volterra and Fredholm examples are given as numerical illustration;
- special case for comparing our proposed with the classical HPM method.

2. SOR Homotopy perturbation method

2.1. Homotopy perturbation method

To illustrate the basic concept of Homotopy perturbation method, consider the following non-linear functional equation

$$\mathcal{A}(u) - f(r) = 0, \quad r \in \Omega, \tag{1}$$

with boundary conditions;

$$B\left(u, \frac{\partial u}{\partial n}\right) = 0, \quad r \in \Gamma,$$

where $\mathcal{A}$ is a general functional operator, B is a boundary operator, $f(r)$ is a known analytic function, and Γ is the boundary of the domain Ω. Generally speaking the operator $\mathcal{A}$ can be divided into two parts L and N, where L is a linear, while N is a nonlinear operator. Therefore, (1) can be rewritten as follows

$$L(u) + N(u) - f(r) = 0.$$

We construct a homotopy $v(r,p)\colon \Omega \times [0,1] \to R$ which satisfies

$$H(v,p) = (1-p)\left[L(v) - L(u_0)\right] + p\left[N(v) - f(r)\right] = 0, \tag{2}$$

or

$$H(v,p) = L(v) - L(u_0) + pL(u_0) + p\left[N(v) - f(r)\right] = 0, \tag{3}$$

where $p \in [0,1]$ is an embedding parameter, and u_0 is an initial approximation for the solution of (1), which satisfies the boundary conditions. According to HPM, we can first use the embedding parameter p as a small parameter, and assume that the solution of (3) can be written as a power series in p:

$$v = v_0 + v_1 p + v_2 p^2 + \ldots = \sum_{i=0}^{\infty} v_i p^i. \tag{4}$$

Considering $p = 1$, the approximation solution of (1) will be obtained as follow

$$u = \lim_{p \to 1} v = v_0 + v_1 + v_2 + \ldots. \tag{5}$$

The series (5) is convergent for most cases, however, the convergent rate depends upon the nonlinear operator $\mathcal{A}(v)$ [3].

2.2. SOR Homotopy perturbation method

Let us rewrite (3) in the following

$$L(v) - L(u_0) = p\left[f(r) - L(u_0) - N(v)\right]. \tag{6}$$

Substituting (4) into (6) leads to

$$L\Big(\sum_{i=0}^{\infty} v_i p^i\Big) - L(u_0) = p\left[f(r) - L(u_0) - N\Big(\sum_{i=0}^{\infty} v_i p^i\Big)\right].$$

So,

$$\sum_{i=0}^{\infty} L(v_i) p^i - L(u_0) = p\left[f(r) - L(u_0) - N\Big(\sum_{i=0}^{\infty} v_i p^i\Big)\right]. \tag{7}$$

According to Maclaurin expansion of $N\big(\sum_{i=0}^{\infty} v_i p^i\big)$ with respect to p, we have

$$N\Big(\sum_{i=0}^{\infty} v_i p^i\Big) = \sum_{n=0}^{\infty}\left[\frac{1}{n!}\frac{\partial n}{\partial p^n} N\Big(\sum_{i=0}^{\infty} v_i p^i\Big)\right]_{p=0} p^i.$$

From [14], we get

$$\left[\frac{\partial n}{\partial p^n} N\Big(\sum_{i=0}^{\infty} v_i p^i\Big)\right]_{p=0} = \left[\frac{\partial n}{\partial p^n} N\Big(\sum_{i=0}^{n} v_i p^i\Big)\right]_{p=0}.$$

Then

$$N\Big(\sum_{i=0}^{\infty} v_i p^i\Big) = \sum_{n=0}^{\infty} \left[\frac{1}{n!}\frac{\partial n}{\partial p^n} N\Big(\sum_{i=0}^{n} v_i p^i\Big)\right]_{p=0} p^i.$$

We set

$$H_n(v_0, v_1, \ldots, v_n) = \left[\frac{1}{n!}\frac{\partial n}{\partial p^n} N\Big(\sum_{i=0}^{n} v_i p^i\Big)\right]_{p=0}, \quad n = 0, 1, 2, \ldots,$$

where H_n's are the so-called He's polynomials [14]. Then

$$N\Big(\sum_{i=0}^{\infty} v_i p^i\Big) = \sum_{i=0}^{\infty} H_i p^i. \tag{8}$$

Substituting (8) into (7), we drive

$$\sum_{i=0}^{\infty} L(v_i)\, p^i - L(u_0) = p\left[f(r) - L(u_0) - \sum_{i=0}^{\infty} H_i p^i\right].$$

By equating the terms with the identical powers in p:

$$\begin{cases} p^0: & L(v_0) - L(u_0) = 0, \\ p^1: & L(v_1) = f(r) - L(u_0) - H_0, \\ p^2: & L(v_2) = -H_1, \\ \ldots & \ldots, \\ p^{n+1}: & L(v_{n+1}) = -H_n, \end{cases}$$

So, we derive,

$$\begin{cases} v_0 = u_0, \\ v_1 = L^{-1}[f(r)] - u_0 - L^{-1}(H_0), \\ v_2 = -L^{-1}(H_1), \\ \ldots, \\ v_{n+1} = -L^{-1}(H_n). \end{cases}$$

Let introduce a parameter ω, and define the sequence $\overline{v}_n$ as follow

$$\begin{cases} \overline{v}_0 = u_0, \\ \overline{v}_1 = \omega\left(L^{-1}[f(r)] - u_0 - L^{-1}(H_0)\right), \\ \overline{v}_2 = \omega\left(-L^{-1}(H_1)\right) + (1-\omega)v_1, \\ \ldots, \\ \overline{v}_{n+1} = \omega\left(-L^{-1}(H_n)\right) + (1-\omega)v_n, \end{cases} \tag{9}$$

then

$$\begin{aligned} \overline{v}_1 + \overline{v}_2 + \ldots + \overline{v}_{n+1} &= \omega\left(L^{-1}[f(r)] - u_0 - L^{-1}(H_0)\right) + \omega\left(-L^{-1}(H_1)\right) + (1-\omega)v_1 \\ &\quad + \ldots + \omega(-L^{-1}(H_n)) + (1-\omega)v_n \\ &= \omega v_1 + (1-\omega)v_1 + \omega v_2 + \ldots + (1-\omega)v_n + \omega v_{n+1} \\ &= v_1 + v_2 + \ldots + \omega v_{n+1} \\ &= \omega\sum_{i=1}^{n+1} v_i + (1-\omega)\sum_{i=1}^{n} v_i. \end{aligned}$$

So,

$$\sum_{i=0}^{n+1} \overline{v}_i = \omega\sum_{i=0}^{n+1} v_i + (1-\omega)\sum_{i=0}^{n} v_i.$$

As $n \to \infty$, we get $\sum_{i=0}^{\infty} \overline{v}_i = \omega\sum_{i=0}^{\infty} v_i + (1-\omega)\sum_{i=0}^{\infty} v_i = \sum_{i=0}^{\infty} v_i$.

If the series $\sum_{i=0}^{\infty} v_i$ is convergent then the series $\sum_{i=0}^{\infty} \overline{v}_i$ is also convergent.

Remark 1. If $\omega = 1$ then the proposed method is the Homotopy perturbation method.

Theorem 1. *Homotopy perturbation method used the solution of (1) is equivalent to determining the following sequence*

$$s_n = v_1 + \ldots + v_n,$$
$$s_0 = 0,$$

where

$$s_{n+1} = -L^{-1}N_n(s_n + v_0) - u_0 + L^{-1}(f(r)),$$

and

$$N_n\Big(\sum_{i=0}^{n} v_i\Big) = \sum_{i=0}^{n} H_i, \quad n = 0, 1, 2, \ldots.$$

Proof. See [15]. ■

Let define the following sequence

$$sr_n = \overline{v}_1 + \ldots + \overline{v}_n,$$
$$sr_0 = 0,$$

where

$$sr_{n+1} = \omega\left(-L^{-1}N(sr_n + \overline{v}_0) - u_0 + L^{-1}(f(r))\right) + (1-\omega)s_n, \tag{10}$$

and

$$N\Big(\sum_{i=0}^{n} \overline{v}_i\Big) = \sum_{i=0}^{n} H_i, \quad n = 0, 1, 2, \ldots.$$

Theorem 2. *Let $\mathbb{B}$ be Banach space,*

1. *$\sum_{j=0}^{\infty} \overline{v}_j$ obtained by (9), converges to $s \in \mathbb{B}$, if*

$$\exists (0 \leqslant \mu < 1),\ s.t\ (\forall u, v \in \mathbb{B}, \left\|L^{-1}N(u) - L^{-1}N(v)\right\| \leqslant \mu \left\|u - v\right\|) \text{ and } 0 < \omega < \frac{2}{1+\mu}.$$

2. *$sr = \sum_{j=1}^{\infty} \overline{v}_n$, satisfies in*

$$sr = -L^{-1}N(sr + v_0) - u_0 + L^{-1}(f(r)).$$

Proof.

1. We have

$$\begin{aligned}
\|sr_{n+1} - sr_n\| &= \big\|\omega(-L^{-1}N(sr_n + \overline{v}_0) - u_0 + L^{-1}(f(r))) + (1-\omega)sr_n \\
&\quad - \omega(-L^{-1}N(sr_{n-1} + \overline{v}_0) - u_0 + L^{-1}(f(r))) + (1-\omega)sr_{n-1}\big\| \\
&\leqslant \big\|\omega(-L^{-1}N(sr_n + \overline{v}_0) + L^{-1}N(sr_{n-1} + \overline{v}_0)) + (1-\omega)(sr_n - sr_{n-1})\big\| \\
&\leqslant \omega\big\|\big(-L^{-1}N(sr_n + \overline{v}_0) + L^{-1}N(sr_{n-1} + \overline{v}_0)\big)\big\| + |1-\omega|\big\|sr_n - sr_{n-1}\big\| \\
&\leqslant \omega\mu\big\|sr_n - sr_{n-1}\big\| + |1-\omega|\big\|sr_n - sr_{n-1}\big\| \\
&\leqslant (\omega\mu + |1-\omega|)\big\|sr_n - sr_{n-1}\big\|.
\end{aligned}$$

Let $\rho = \omega\mu + |1-\omega|$, then

$$\begin{aligned}
\big\|sr_{n+1} - sr_n\big\| &\leqslant \rho\big\|sr_n - sr_{n-1}\big\| \\
&\leqslant \rho^2\big\|sr_{n-1} - sr_{n-2}\big\| \\
&\leqslant \rho^3\big\|sr_{n-2} - sr_{n-3}\big\| \\
&\ldots \\
&\leqslant \rho^n\big\|sr_1 - sr_0\big\|.
\end{aligned}$$

If $0 < \omega < 1$, then $\rho = \omega\mu + 1 - \omega = 1 - \omega(1-\mu) < 1$.
If $\omega > 1$, then $\rho = \omega\mu + \omega - 1 = \omega(1+\mu) - 1$. So, $\omega < \frac{2}{1+\mu}$ implies $\rho < 1$.
For any n, $m \in \mathbb{N}$, $n \geqslant m$, we derive

$$\begin{aligned}\|sr_n - sr_m\| &\leqslant \|sr_n - sr_{n-1} + sr_{n-1} - sr_{n-2} + \ldots + sr_{m+1} - sr_m\| \\ &\leqslant \|sr_n - sr_{n-1}\| + \|sr_{n-1} - sr_{n-2}\| + \ldots + \|sr_{m+1} - sr_m\| \\ &\leqslant \rho^n \|sr_1 - sr_0\| + \rho^{n-1} \|sr_1 - sr_0\| + \ldots + \rho^{m+1} \|sr_1 - sr_0\| \\ &\leqslant (\rho^n + \rho^{n-1} + \ldots + \rho^{m+1}) \|sr_1 - sr_0\| \\ &\leqslant (\rho^{m+1} + \ldots + \rho^n + \ldots) \|sr_1 - sr_0\| \\ &\leqslant \rho^{m+1}(1 + \rho + \ldots + \rho^n + \ldots) \|sr_1 - sr_0\| \\ &\leqslant \frac{\rho^{m+1}}{1-\rho} \|sr_1 - sr_0\|.\end{aligned}$$

So,

$$\lim_{n,m\to\infty} \|sr_n - sr_m\| = 0.$$

Then $\{sr_n\}$ is the Cauchy sequence in a Banach space and so it is convergent, i.e.,

$$\exists sr \in B, \text{ s.t } \lim_{n\to\infty} sr_n = \sum_{j=1}^{\infty} \overline{v}_j = sr.$$

2. From (10), we have

$$\begin{aligned}\lim_{n\to\infty} sr_{n+1} &= -L^{-1} \lim_{n\to\infty} N(sr_n + \overline{v}_0) - u_0 + L^{-1}(f(r)) \\ &= -L^{-1} \lim_{n\to\infty} N\Big(\sum_{j=0}^{n} \overline{v}_j\Big) - u_0 + L^{-1}(f(r)), \\ sr &= -L^{-1} \lim_{n\to\infty} \sum_{j=0}^{n} H_j - u_0 + L^{-1}(f(r)) \\ &= -L^{-1} \sum_{j=0}^{\infty} H_j - u_0 + L^{-1}(f(r)).\end{aligned}$$

But by (8) for $p = 1$, we drive

$$\sum_{j=0}^{\infty} H_j = N\Big(\sum_{j=0}^{\infty} \overline{v}_j\Big).$$

So,

$$\begin{aligned}sr &= -L^{-1} N\Big(\sum_{j=0}^{\infty} \overline{v}_j\Big) - u_0 + L^{-1}(f(r)), \\ sr &= -L^{-1} N(sr + v_0) - u_0 + L^{-1}(f(r)).\end{aligned}$$

■

3. Illustration examples

Example 1. Consider the first order nonlinear ordinary differential equation:

$$u'(x) + u^2(x) = 1, \quad |x| < 1. \tag{11}$$

Here, $\mathcal{A}(u) = u' + u^2$ and $f(x) = 0$. The operator $\mathcal{A}$ can be divided into two parts L and N, where $L(u) = u'$ and $N(u) = u^2$.

In this case, equation (2) becomes:

$$v'(x) - u_0'(x) + p u_0'(x) + p\big[v^2(x)\big] = 0, \quad p \in [0, 1].$$

Assume the solution of the above equation can be written as given in equation (4). By substituting this solution into the above equation one can have:

$$\sum_{i=0}^{\infty} p^i v_i'(x) - u_0'(x) + p u_0'(x) + p\Big(\sum_{i=0}^{\infty} p^i v_i(x)\Big)^2 = 0.$$

By equating the terms with identical powers of p one can have:

$$\begin{cases} p^0: & v_0' - u_0' = 0, \\ p^1: & v_1' + u_0' + v_0^2(x) = 0, \\ p^2: & v_2' + 2v_0(x)v_1(x) = 0, \\ p^3: & v_3' + 2v_0(x)v_2(x) + v_2^2(x) = 0, \\ \ldots & \ldots . \end{cases}$$

For simplicity, let $v_0(x) = u_0(x) = 1$ the initial approximation of the differential equation (11), then $v_1'(x) + 1 = 0$ and this implies that: $v_1(x) = -x$, also $v_2'(x) - 2x = 0$ which implies $v_2(x) = x^2$. Continuing in this manner, one can have:

$$v_n(x) = (-1)^n x^n, \quad n = 0, 1, 2, \ldots .$$

Now, let $\overline{v}_0 = v_0 = u_0$ and for $n = 1, 2, 3, \ldots$

$$\begin{aligned} \overline{v}_n(x) &= \omega v_n(x) + (1-\omega)v_{n-1}(x) \\ &= (-1)^n \left(\omega x^n - (1-\omega)x^{n-1}\right). \end{aligned}$$

Then

$$\begin{aligned} \sum_{n=1}^{\infty} \overline{v}_n(x) &= \sum_{n=1}^{\infty}(-1)^n \left(\omega x^n - (1-\omega)x^{n-1}\right) \\ &= \omega \sum_{n=1}^{\infty}(-1)^n x^n + (1-\omega)\sum_{n=1}^{\infty}(-1)^{n-1}x^{n-1}. \end{aligned}$$

So,

$$\begin{aligned} \overline{u} &= \sum_{n=0}^{\infty} \overline{v}_n(x) = \omega \sum_{n=0}^{\infty}(-1)^n x^n + (1-\omega)\sum_{n=0}^{\infty}(-1)^n x^n \\ &= \sum_{n=0}^{\infty}(-1)^n x^n = \frac{1}{1+x}, \end{aligned}$$

which is the exact solution of the ordinary differential equation (11).

3.1. Fredholm integral equation of the second kind

Now we consider the Fredholm integral equation of the second kind in general case,

$$u(x) = f(x) + \lambda \int_a^b k(x,t)\, u(t)\, dt, \tag{12}$$

where $k(x,t)$ is the kernel of the integral equation. In view of (2)

$$(1-p)\left[u(x) - f(r)\right] + p\left[u(x) - f(x) - \lambda \int_a^b k(x,t)\, u(t)\, dt\right] = 0,$$

or

$$u(x) = f(x) + p\,\lambda \int_a^b k(x,t)\, u(t)\, dt. \tag{13}$$

Substituting (4) into (13), and equating the terms with identical powers of p, we have

$$\begin{cases} p^0: & u_0 = f(x), \\ p^1: & u_1 = \lambda \displaystyle\int_a^b k(x,t)(u_0)\, dt, \\ p^2: & u_2 = \lambda \displaystyle\int_a^b k(x,t)(u_1)\, dt, \\ p^3: & u_3 = \lambda \displaystyle\int_a^b k(x,t)(u_2)\, dt, \\ \ldots, & \end{cases}$$

therefore, we obtain iteration formula for (12) as follow:

$$\begin{cases} u_0(x) = f(x), \\ u_m(x) = \lambda \displaystyle\int_a^b k(x,t) u_{m-1}(t)\, dt, \quad m > 0. \end{cases}$$

Then

$$\begin{cases} \overline{u}_0(x) = f(x), \\ \overline{u}_m(x) = \omega\lambda u_m(x) + (1-\omega) u_{m-1}(x), \quad m > 0. \end{cases} \tag{14}$$

According to (14) we define partial sum as follow

$$\begin{cases} sr_0(x) = f(x), \\ sr_n(x) = \displaystyle\sum_{i=0}^{n} \overline{u}_i(x), \end{cases} \tag{15}$$

in view of (14)–(15) we have

$$\begin{cases} s_0(x) = f(x), \\ s_n(x) = f(x) + \lambda \displaystyle\int_a^b k(x,t)\, s_m(t)\, dt. \end{cases}$$

Theorem 3. *Consider the iteration scheme*

$$\begin{cases} sr_0(x) = f(x), \\ sr_{n+1}(x) = \omega \left(f(x) + \lambda \displaystyle\int_a^b k(x,t)\, sr_n(t)\, dt \right) + (1-\omega) sr_n \end{cases}$$

for $n = 0, 1, 2, \ldots$ to construct a sequence of successive iterations $sr_n(x)$ to the solution of (12). In addition, let

$$\int_a^b \int_a^b k^2(x,t)\, dx\, dt = B^2 < \infty,$$

and assume that $f(x) \in L^2(a,b)$. Then, if $\mu = |\lambda|\, B < 1$ and $0 < \omega < \frac{2}{1+\mu}$, the above iteration converges in the norm of $L^2(a,b)$ to the solution of (12).

Proof. By Theorem 2, we have

$$\begin{aligned} \big\| sr_{n+1}(x) - sr_n(x) \big\| &= \Bigg\| \omega \left(f(x) + \lambda \int_a^b k(x,t)\, sr_n(t)\, dt \right) + (1-\omega) sr_n(x) \\ &\qquad - \left[\omega \left(f(x) + \lambda \int_a^b k(x,t)\, sr_{n-1}(t)\, dt \right) + (1-\omega) sr_{n-1}(x) \right] \Bigg\| \\ &\leqslant \left\| \omega\lambda \int_a^b k(x,t) \big(sr_n(t) - sr_{n-1}(t) \big) dt + (1-\omega) \big(sr_n(x) - sr_{n-1}(x) \big) \right\| \\ &\leqslant \omega\, |\lambda| \left\| \int_a^b k(x,t)\, dt \right\| \big\| sr_n(x) - sr_{n-1}(x) \big\| + |1-\omega|\, \big\| sr_n(x) - sr_{n-1}(x) \big\| \\ &\leqslant \left(\omega\, |\lambda| \left\| \int_a^b k(x,t)\, dt \right\| + |1-\omega| \right) \big\| sr_n(x) - sr_{n-1}(x) \big\|. \end{aligned}$$

Let $\mu = |\lambda| \left\| \int_a^b k(x,t)\, dt \right\|$, then $\omega\, |\lambda| \left\| \int_a^b k(x,t)\, dt \right\| + |1-\omega| < 1$, if $0 < \omega < \frac{2}{1+\mu}$ and the series $\{sr_n\}$ is convergent and it converges to the solution of (12). ■

Example 2. Consider the integral equation

$$u(x) = \sqrt{x} + \lambda \int_0^1 x\, t\, u(t)\, dt, \tag{16}$$

its iterations formula, by our methods, reads

$$sr_{n+1}(x) = \omega \left(\sqrt{x} + \lambda \int_0^1 (x\, t)\, sr_n(t)\, dt \right) + (1-\omega) sr_n(x), \tag{17}$$

and

$$u_0(x) = \sqrt{x},$$

and in view of (13), we obtained

$$u(x) = \sqrt{x} + p\lambda \int_0^1 x\,t\,u(t)\,dt,$$

Substituting (2) into (16), we have the following results

$$\begin{cases} p^0: & u_0(x) = \sqrt{x}, \\ p^1: & u_1(x) = \lambda \int_0^1 x\,t\,\sqrt{t}\,dt = \dfrac{2\lambda x}{5}, \\ p^2: & u_2(x) = \lambda \int_0^1 x\,t\,\dfrac{2\lambda t}{5}\,dt = \dfrac{2\lambda^2}{15}x, \\ p^3: & u_3(x) = \lambda \int_0^1 x\,t\,\dfrac{2\lambda^2}{15}t\,dt = \dfrac{2\lambda^3}{45}x, \\ \dots & \dots \end{cases}$$

and

$$\begin{cases} \overline{u}_0(x) = \sqrt{x}, \\ \overline{u}_1(x) = \omega u_1(x) \\ \overline{u}_i(x) = \omega u_1(x) + (1-\omega)u, \quad i = 2, 3, \dots. \end{cases}$$

Continuing this way ad infinitum, we obtain

$$\begin{aligned} sr_n(x) &= \omega\left(\sqrt{x} + \left[\frac{2}{5.3^0}\lambda + \frac{2}{5.3^1}\lambda^2 + \ldots + \frac{2}{5.3^{n-1}}\lambda^n\right]\right) \\ &\quad + (1-\omega)\left(\sqrt{x} + \left[\frac{2}{5.3^0}\lambda + \frac{2}{5.3^1}\lambda^2 + \ldots + \frac{2}{5.3^{n-2}}\lambda^{n-1}\right]\right) \\ &= \omega\left(\sqrt{x} + \frac{6}{5}\sum_{i=1}^{n}\left(\frac{\lambda}{3}\right)^i x\right) + (1-\omega)\left(\sqrt{x} + \frac{6}{5}\sum_{i=1}^{n-1}\left(\frac{\lambda}{3}\right)^i x\right). \end{aligned}$$

The above sequence is convergent if $|\lambda| < 3$.

Note that by Theorem 1 we have

$$\int_a^b \int_a^b k^2(x,t)\,dx\,dt = \int_0^1 \int_0^1 (x\,t)^2\,dx\,dt = \frac{1}{9} = B^2.$$

Then if $|\lambda| < 3$ (17) is convergent.

3.2. Volterra integral equations of the second kind

We consider the Volterra integral equations of the second kind in the form

$$u(x) = f(x) + \lambda \int_a^x k(x,t)\,u(t)\,dt,$$

where $K(x,t)$ is the kernel of the integral equation. As in the case of the Fredholm integral equation, we can use SOR Homotopy perturbation method to solve Volterra integral equations of the second kind.

However, there is one important difference: if $K(x,t)$ and $f(x)$ are real and continuous, then the series converges for all values of λ (see [16]).

Example 3. Consider the integral equation

$$u(x) = x + \lambda \int_0^x (x-t)\,u(t)\,dt,$$

its iteration formula reads

$$s_{n+1}(x) = x + \lambda \int_0^x (x-t)\,s_n(t)\,dt,$$

and

$$u_0(x) = x,$$

in view of (12), we obtained

$$u(x) = x + p\lambda \int_0^x (x-t)\, u(t)\, dt, \tag{18}$$

Substituting (2) into (18), we have the following results

$$\begin{cases} p^0: & u_0(x) = x, \\ p^1: & u_1(x) = \lambda \displaystyle\int_0^1 (x-t)\, t\, dt = \frac{x^3}{3!}\lambda, \\ p^2: & u_2(x) = \lambda \displaystyle\int_0^1 (x-t)\, \frac{t^3}{3!}\lambda\, dt = \frac{x^5}{5!}\lambda^2, \\ p^3: & u_3(x) = \lambda \displaystyle\int_0^1 (x-t)\, \frac{t^5}{5!}\lambda^2 dt = \frac{x^7}{7!}\lambda^3, \\ \dots & \dots \end{cases}$$

and

$$\begin{cases} \overline{u}_0(x) = x, \\ \overline{u}_1(x) = \omega u_1(x) \\ \overline{u}_i(x) = \omega u_1(x) + (1-\omega)u, \quad i = 2, 3, \dots. \end{cases}$$

Continuing this way ad infinitum, we obtain

$$sr_n(x) = \omega \left(\sum_{i=0}^{n} \lambda^i \frac{x^{2i-1}}{(2i-1)!} \right) + (1-\omega) \left(\sum_{i=0}^{n-1} \lambda^i \frac{x^{2i-1}}{(2i-1)!} \right).$$

The above sequence is convergent for all λ.

Example 4. Consider the following integro-differential equation

$$u''(x) = -1 + \lambda \int_0^x (x-t)\, u(t)\, dt,$$

which is equivalent to

$$u(x) = 1 - \frac{x^2}{2!} + \frac{\lambda}{3!} \int_0^x (x-t)^3 u(t)\, dt,$$

by SOR Homotopy perturbation method, its iteration formula reads

$$sr_{n+1}(x) = \omega \left(1 - \frac{x^2}{2!} + \frac{\lambda}{3!} \int_0^x (x-t)^3 s_n(t)\, dt \right) + (1-\omega) sr_n(x),$$

in view of (13), we obtained

$$u(x) = 1 - \frac{x^2}{2!} + \frac{\lambda}{3!} p \int_0^x (x-t)^3 u(t)\, dt, \tag{19}$$

and

$$u_0(x) = 1 - \frac{x^2}{2!}.$$

Substituting (2) into (19), we have the following results

$$\begin{cases} p^0: & u_0(x) = 1 - \dfrac{x^2}{2!}, \\ p^1: & u_1(x) = \dfrac{\lambda}{3!} \displaystyle\int_0^x (x-t)^3 \left(1 - \frac{t^2}{2!}\right) dt = \left(\frac{x^4}{4!} - \frac{x^3}{6!}\right)\lambda, \\ p^2: & u_2(x) = \dfrac{\lambda}{3!} \displaystyle\int_0^x (x-t) \left(\frac{x^4}{4!} - \frac{x^3}{6!}\right) dt = \left(\frac{x^8}{8!} - \frac{x^{10}}{10!}\right)\lambda^2, \\ p^3: & u_3(x) = \lambda \displaystyle\int_0^1 (x-t) \left(\frac{t^8}{8!} - \frac{t^{10}}{10!}\right) \lambda^2 dt = \left(\frac{x^{12}}{12!} - \frac{x^{14}}{14!}\right)\lambda^3, \\ \dots & \dots \end{cases}$$

and

$$\begin{cases} \overline{u}_0(x) = x, \\ \overline{u}_1(x) = \omega u_1(x), \\ \overline{u}_i(x) = \omega u_1(x) + (1-\omega)u, \ i = 2, 3, \dots. \end{cases}$$

Continuing this way ad infinitum, we obtain

$$sr_n(x) = \omega \left(\sum_{i=0}^{n} \lambda^i \frac{x^{4i}}{(4i)!} + \sum_{i=0}^{n} \lambda^i \frac{x^{4i+2}}{(4i+2)!} \right) + (1-\omega) \left(\sum_{i=0}^{n-1} \lambda^i \frac{x^{4i}}{(4i)!} + \sum_{i=0}^{n} \lambda^i \frac{x^{4i+2}}{(4i+2)!} \right).$$

The above sequence is convergent for all λ. Note, for $\lambda = 1$ and $\omega = 1$ the above sequence converges to $\cos x$ which is the exact solution for

$$u''(x) = -1 + \lambda \int_0^x (x-t)\, u(t)\, dt.$$

3.3. Special case

In the previous sections, we have shown that under some hypothesis the Homotopy perturbation method converges. Also, under the same assumptions plus conditions on the parameter ω, the proposed method converges. Here, we will see cases where the HPM method diverges but under certain conditions our method converges.

To do this, let consider the following Example.

Example 5. Consider the following linear Fredholm integral equation of the second kind:

$$u(x) = e^{3x} - \frac{8}{9}e^3 x - \frac{4}{3}x - 4\int_0^1 x\, t\, u(t)\, dt, \quad 0 \leqslant x \leqslant 1 \tag{20}$$

Here $a = 0$, $b = 1$, $\lambda = -4$, $f(x) = e^{3x} - \frac{8}{9}e^3 x - \frac{4}{3}x$ and $k(x,t) = x\,t$.

Therefore

$$\int_a^b \int_a^b k^2(x,t)\, dx\, dt = \int_0^1 \int_0^1 x^2 t^2 dx\, dt = \frac{1}{9} = B^2 < \infty$$

and

$$|\lambda| = 4 > \frac{1}{B} = 3.$$

So, if we use the Homotopy perturbation method to solve this example, then the solution that is obtained by applying this method may be convergent to the exact solution or may not.

To do this, let

$$u_0(x) = f(x) = e^{3x} - \left(\frac{8}{9}e^3 + \frac{4}{3}\right) x.$$

Hence

$$\begin{aligned} u_1(x) &= -4\int_0^1 x\, t\, f(t)\, dt) = -4\int_0^1 x\, t \left(e^{3t} - \frac{8}{9}e^3 t - \frac{4}{3}t\right) dt \\ &= -4x \left(\int_0^1 t\, e^{3t} dt - \left(\frac{8}{9}e^3 + \frac{4}{3}\right) \int_0^1 t^2 dt \right) \\ &= \frac{4}{3}\left(\frac{2}{9}e^3 - 1\right) x \end{aligned}$$

and

$$\begin{aligned} u_2(x) &= -4\int_0^1 x\, t\, u_1(t)\, dt = -4\int_0^1 x\, t^2 \frac{4}{3}\left(\frac{2}{9}e^3 - 1\right) dt \\ &= -4\frac{4}{3}\left(\frac{2}{9}e^3 - 1\right) x \int_0^1 t^2 dt \\ &= -\left(\frac{4}{3}\right)^2 \left(\frac{2}{9}e^3 - 1\right) x. \end{aligned}$$

By continuing in this manner, one can have:

$$u_i(x) = -4\int_0^1 x\,t\,u_{i-1}(t)\,dt = (-1)^{i+1}\left(\frac{4}{3}\right)^i\left(\frac{2}{9}e^3-1\right)x, \quad i=1,2,\ldots.$$

Thus

$$\begin{aligned} u(x) &= \sum_{i=0}^{\infty} u_i(x) \\ &= e^{3x} - \left(\frac{8}{9}e^3+\frac{4}{3}\right)x + \sum_{i=1}^{\infty}(-1)^{i+1}\left(\frac{4}{3}\right)^i\left(\frac{2}{9}e^3-1\right)x. \end{aligned}$$

Since

$$\sum_{i=1}^{\infty}(-1)^{i+1}\left(\frac{4}{3}\right)^i\left(\frac{2}{9}e^3-1\right)x = \left(\frac{2}{9}e^3-1\right)x\sum_{i=1}^{\infty}(-1)^{i+1}\left(\frac{4}{3}\right)^i.$$

But $\sum_{i=1}^{\infty}(-1)^{i+1}\left(\frac{4}{3}\right)^i$ is an alternating series, that is divergent since $|r| = \frac{4}{3} > 1$. Therefore $\sum_{i=1}^{\infty} u_i(x)$ is divergent.

Now, if we use the SOR Homotopy perturbation method, then we have

$$sr_0(x) = \overline{u}_0(x) = f(x).$$

Hence

$$\begin{aligned} \overline{u}_1(x) &= -4\omega\int_0^1 x\,t\,f(t)\,dt \\ &= \frac{4}{3}\omega\left(\frac{2}{9}e^3-1\right)x \end{aligned}$$

and

$$\begin{aligned} \overline{u}_i(x) &= \omega(-1)^{i+1}\left(\frac{4}{3}\right)^i\left(\frac{2}{9}e^3-1\right)x + (1-\omega)(-1)^i\left(\frac{4}{3}\right)^{i-1}\left(\frac{2}{9}e^3-1\right)x \\ &= \left(-\frac{4}{3}\omega+1-\omega\right)(-1)^i\left(\frac{4}{3}\right)^{i-1}\left(\frac{2}{9}e^3-1\right)x \\ &= \left(-\frac{4}{3}\omega+1-\omega\right)\overline{u}_{i-1}(x), \quad i=2,\ldots. \end{aligned}$$

Then

$$\begin{aligned} sr_i(x) - sr_{i-1}(x) = \overline{u}_i(x) &= \left(-\frac{4}{3}\omega+1-\omega\right)\overline{u}_{i-1}(x) \\ &= \left(-\frac{4}{3}\omega+1-\omega\right)(sr_{i-1}(x) - sr_{i-2}(x)), \quad i=2,\ldots. \end{aligned}$$

If ω is chosen such that $\left|-\frac{4}{3}\omega+1-\omega\right| < 1$ i.e. $\omega < \frac{6}{7}$, then the series is convergent to the exact solution of the equation (20).

4. Conclusion

In this work, we present a SOR Homotopy perturbation method by introducing a parameter ω in order to extend the classical homotopy perturbation method. According to the values of ω, we study the convergence of the method under some assumptions. We take an integral equations and integro-differential equations as example illustration of our proposed method. A particular case is given when our method converges but the classical homotopy perturbation method fails.

Acknowledgements

The authors thanks the referees very much for their constructive suggestions, helpful comments and fast response, which led to significant improvement of the original manuscript of this paper.

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Метод гомотопічних збурень SOR для розв'язування інтегро-диференціальних рівнянь

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У цій статті представлено метод гомотопічних збурень SOR, новий метод декомпозиції шляхом введення параметра ω для розширення класичного методу гомотопічних збурень для розв'язування інтегро-диференціальних рівнянь різних видів. Використовуючи метод гомотопічних збурень SOR та його ітераційну схему, можна дати точний розв'язок або замкнене наближення до розв'язку задачі. Розглянуто збіжність запропонованого методу та наведено деякі ілюстративні приклади із застосуваннями до інтегральних рівнянь Фредгольма та Вольтерра.

Ключові слова: *SOR метод гомотопічних збурень; інтегральне рівняння Фредгольма; інтегральне рівняння Вольтерра.*