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STATISTICAL MODELING OF *κ·q±1* DISCRETE DATA TRANSFORMATION SYSTEMS

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Abstract. A new branching tree model has been proposed for the first time in the direction of increasing degree 2^n (merging in the reverse direction), which coincides with the direction of increasing total stopping time. It has been shown that each time corresponds to a sequence of individual numbers *n(tst)*→∞, the volume of which increases with time. Thus, it is proven that each time corresponds to a finite number of Collatz sequences of the same length. The reason for the formation of a histogram or spectrum *tst(q)* with two peaks has been established. It is shown that the double structure is formed by the regularities of Jacobsthal recurrence numbers at the nodes of the sequences $\{\theta \cdot 2^n\}_{n=0}^{n=\infty}$. It has been established that the graph *tst(q)* with the numbers of active nodes in semi-logarithmic coordinates *tst*, log*m(p)* appears as a straight line, while the graph for the numbers of inactive nodes appears as a scattered spectrum. Based on the established statistical regularities *tst(q)*, a new recurrent model of trivial cycles is proposed.

Keywords: recurrence numbers, recurrence sequences, Jacobsthal numbers, Collatz conjecture, total stopping time, probability, trivial cycle, Collatz sequence, histogram, scattered spectra

Introduction and Problem Statement

In 1976, Richo Terras [1] introduced the concept of the so-called total stopping time (*tst*) as one of the fundamental characteristics of the transformation of natural numbers $q \in N$ by the Collatz algorithm.

$$C_{3q+1} = \quad if \quad q \equiv 0 \mod 2 \quad then \quad \frac{q}{2} \quad else \quad 3q+1,\; N{=}N_{odd} \cup N_{even}\{0\}, \tag{1}$$

with a trivial termination cycle

$$... \rightarrow 4 \rightarrow 2 \rightarrow 1 \rightarrow ... \tag{2}$$

By definition, *tst* is the number of iterations during which an element of the Collatz sequence (*CSq*) reaches the value of one (the single-point attractor). Therefore, *tst* or *N* determines the length of the sequence CSq.

For problem (1), statistical studies of the total stopping time *(tst)* have been conducted relatively recently [2-9]. A significant amount of results is available on electronic pages [10-14]. Lagarias [2] was the first to summarize the results of such studies on the Collatz problem, and a fundamental analysis of recent research was conducted in work [5]. Recently, in [15], the was proposed a statistical model analogous to the distribution of stopping times (*tst*) using the well-known Planck's law: Frequency, $f \leftrightarrow$ Photon frequency.

In this work, the statistical problem of transforming $kq\pm1$ integer discrete data $q \in N$ for a general type of transformations is investigated:

$$C^{\pm}_{\kappa q\pm1} = \quad if \quad q \equiv 0 \mod 2 \quad then \quad \frac{q}{2} \quad else \quad \kappa \cdot q \pm 1, \; \kappa \in \mathrm{N_{odd}}. \tag{3}$$

Other results of the authors' research are presented in works [16-20].

Objectives and Problems of Research

Let's consider the task of transforming the set of natural numbers $q \in N$ into a set of parameterized θ sequences $\left\{\theta \cdot 2^n\right\}_{n=0}^{n=\infty}$ by powers of two 2^n:

$$\begin{matrix} 1\cdot2^0 & 3\cdot2^0 & 5\cdot2^0 & 7\cdot2^0 & \dots & \theta\cdot2^0 & \dots \\ 1\cdot2^1 & 3\cdot2^1 & 5\cdot2^1 & 7\cdot2^1 & \dots & \theta\cdot2^1 & \dots \\ 1\cdot2^2 & 3\cdot2^2 & 5\cdot2^2 & 7\cdot2^2 & \dots & \theta\cdot2^2 & \dots \\ 1\cdot2^3 & 3\cdot2^3 & 5\cdot2^3 & 7\cdot2^3 & \dots & \theta\cdot2^3 & \dots \\ 1\cdot2^4 & 3\cdot2^4 & 5\cdot2^4 & 7\cdot2^4 & \dots & \theta\cdot2^4 & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ 1\cdot2^n & 3\cdot2^n & 5\cdot2^n & 7\cdot2^n & 9\cdot2^n & \theta\cdot2^n & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots \end{matrix} \quad \theta \in \mathrm{N_{odd}},\; n \in \mathrm{N} \cup \{0\}, \tag{4}$$

An illustration of the structuring (1) is shown in Fig. 1. In the sequences with the θ_3 parameter, which is a multiple of three, all the cells are colored the same because, as will be shown later, in the model of nodes with Jacobsthal recurrent numbers [16], on sequences $\left\{\theta_3 \cdot 2^n\right\}_{n=0}^{n=\infty}$ branching points in $n \to \infty$ direction (merging in $n \to 0$ direction) of other sequences are absent. The so-called Jacobsthal tree is formed by the branching of other sequences at nodes with parameters $\theta \neq \theta_3$ (Jacobsthal numbers) that are not multiples of three. Therefore, the sequences $\left\{\theta \neq \theta_3 \cdot 2^n\right\}_{n=0}^{n=\infty}$ are colored every other cell in the columns.

n	$1\cdot2^n$	$3\cdot2^n$	$5\cdot2^n$	$7\cdot2^n$	$9\cdot2^n$	$11\cdot2^n$	$13\cdot2^n$
0	$1\cdot2^0$	$3\cdot2^0$	$5\cdot2^0$	$7\cdot2^0$	$9\cdot2^0$	$11\cdot2^0$	$13\cdot2^0$
1	$1\cdot2^1$	$3\cdot2^1$	$5\cdot2^1$	$7\cdot2^1$	$9\cdot2^1$	$11\cdot2^1$	$13\cdot2^1$
2	$1\cdot2^2$	$3\cdot2^2$	$5\cdot2^2$	$7\cdot2^2$	$9\cdot2^2$	$11\cdot2^2$	$13\cdot2^2$
3	$1\cdot2^3$	$3\cdot2^3$	$5\cdot2^3$	$7\cdot2^3$	$9\cdot2^3$	$11\cdot2^3$	$13\cdot2^3$
4	$1\cdot2^4$	$3\cdot2^4$	$5\cdot2^4$	$7\cdot2^4$	$9\cdot2^4$	$11\cdot2^4$	$13\cdot2^4$
5	$1\cdot2^5$	$3\cdot2^5$	$5\cdot2^5$	$7\cdot2^5$	$9\cdot2^5$	$11\cdot2^5$	$13\cdot2^5$
6	$1\cdot2^6$	$3\cdot2^6$	$5\cdot2^6$	$7\cdot2^6$	$9\cdot2^6$	$11\cdot2^6$	$13\cdot2^6$
7	$1\cdot2^7$	$3\cdot2^7$	$5\cdot2^7$	$7\cdot2^7$	$9\cdot2^7$	$11\cdot2^7$	$13\cdot2^7$
8	$1\cdot2^8$	$3\cdot2^8$	$5\cdot2^8$	$7\cdot2^8$	$9\cdot2^8$	$11\cdot2^8$	$13\cdot2^8$
9	$1\cdot2^9$	$3\cdot2^9$	$5\cdot2^9$	$7\cdot2^9$	$9\cdot2^9$	$11\cdot2^9$	$13\cdot2^9$
10	$1\cdot2^{10}$	$3\cdot2^{10}$	$5\cdot2^{10}$	$7\cdot2^{10}$	$9\cdot2^{10}$	$11\cdot2^{10}$	$13\cdot2^{10}$
11	$1\cdot2^{11}$	$3\cdot2^{11}$	$5\cdot2^{11}$	$7\cdot2^{11}$	$9\cdot2^{11}$	$11\cdot2^{11}$	$13\cdot2^{11}$
12	$1\cdot2^{12}$	$3\cdot2^{12}$	$5\cdot2^{12}$	$7\cdot2^{12}$	$9\cdot2^{12}$	$11\cdot2^{12}$	$13\cdot2^{12}$

Fig. 1. Illustration of the structuring of the N as a set of parameterized θ sequences

Binary-based sequences $\{\theta\cdot 2^n\}_{n=0}^{n=\infty}$ are used in algorithms of type (1) and (2) to halve even numbers $q/2$ until they take on an odd value in the direction of decreasing $n\to 0$ the power n, and to double any number in the reverse $n\to\infty$ direction. Therefore, structuring the set $q\in N$ in the form of (1) allows the problem (3) to be reduced to justifying the rules of superposition between sequences $\{\theta\cdot 2^n\}_{n=0}^{n=\infty}$ with the set of parameters $\theta\in N_{odd}$,, namely the rules of their merging ($n\to 0$) and branching ($n\to\infty$) at nodes with Jacobsthal numbers.

The role of Jacobsthal numbers in the Collatz problem $3q\pm 1$ has been studied in detail in works [16-20], and they were first brought to attention in this context in [21]. It has been established that the principle of superposition of sequences (4) can be implemented in the model of nodes with Jacobsthal recurrent integers in a closed form:

$$J^{\pm}_{\kappa,\theta,n}=\frac{\theta\cdot 2^n\pm(-1)^n}{\kappa},\ \kappa=1,3,5,...\in N_{odd}, \tag{5}$$

Below are the numbers for $\kappa=1\div 9$ are given in the table:

Table 1.

Numbers $Jm_{\kappa,1,n}=\dfrac{\theta\cdot 2^n-(-1)^n}{\kappa}$ **and** $Jp_{\kappa,1,n}=\dfrac{\theta\cdot 2^n+(-1)^n}{\kappa}$

$Jp_{9,1,n}$	$Jp_{7,1,n}$	$Jp_{5,1,n}$	$Jp_{3,1,n}$	$Jp_{1,1,n}$	2^n	$Jm_{1,1,n}$	$Jm_{3,1,n}$	$Jm_{5,1,n}$	$Jm_{7,1,n}$	$Jm_{9,1,n}$
			2731	8193	2^{13}	8191	-			-
455	585	819	1365	4095	2^{12}	4097	-			-
			683	2049	2^{11}	2047	-			-
			341	*1023*	$\mathbf{2^{10}}$	1025	-	205		-
57			171	513	2^{9}	511	-		73	-
		51	85	*255*	$\mathbf{2^{8}}$	257	-			-
			43	129	$\mathbf{2^{7}}$	127	-	-		-
7	9		21	***63***	$\mathbf{2^{6}}$	65	-	13		-
			11	**33**	2^{5}	31	-			-
		3	5	*15*	$\mathbf{2^{4}}$	17	-			-
1			3	9	2^{3}	7	-		1	-
			1	*3*	$\mathbf{2^{2}}$	5`	**-**	**1**		-
			1	3	2^{1}	1	-			-
0	0	0	0	*0*	$\mathbf{2^{0}}$	2	-	0		-

Here, the numbers are calculated for the sequence parameter $\theta=1$, as in problem (2), the formation of Collatz sequences (CS_q) of any number $q\in N$ is completed with the participation of the root sequence $\{1\cdot 2^n\}_{n=0}^{n=\infty}$. In Table 1, the integers $J^{+}_{\kappa,1,n}=Jp_{\kappa,1,n}$ and $J^{-}_{\kappa,1,n}=Jm_{\kappa,1,n}$ are given. The numbers $Jm(p)_{\kappa,\theta,n}$ are fractional if κ and θ are both multiples of three simultaneously; otherwise, if only one of the parameters κ or θ is a multiple of three, then $Jm_{\kappa,\theta,n}$ is fractional. As follows from the first column of Table 2, the adjacent numbers $J^{\pm}_{\kappa=1,3,5,\theta,n}$ are related to each other by second-order recurrence formulas, whereas the numbers $J^{\pm}_{\kappa\geq 7,\theta,n}$ are related by first-order formulas.

Table 2.

Recurrence formulas for numbers $J^{\pm}_{\kappa,\theta,n}$ and $m(p)_{\kappa,\theta,n}$

κ	$J_{\kappa,\theta,n}$	$m(p)_{\kappa,\theta,n}$
$\kappa=1$	$J_{1,\theta,n+2}=J_{3,\theta,n+1}+2\cdot J_{1,\theta,n}$, $J=Jm(p)$	$m(p)_{1,\theta,n+1}=2m(p)_{1,\theta,n}\pm 1$,
$\kappa=3$	$J_{3,\theta,n+2}=J_{3,\theta,n+1}+2\cdot J_{3,\theta,n}$, $J=Jp$	$m(p)_{3,\theta,n+2}=4m(p)_{3,\theta,n}\pm 1$
$\kappa=5$	$J_{5,\theta,n+4}=3\cdot J_{5,\theta,n+2}+4\cdot J_{5,\theta,n}$, $J=Jm\cup Jp$	$m(p)_{5,\theta,n+4}=16m(p)_{5,\theta,n}\pm 3$
$\kappa=7$	$J_{7,\theta,n+3}=8\cdot J_{7,\theta,n}+1$, $J=Jm\cup Jp$	$m(p)_{7,\theta,n+6}=64m(p)_{7,\theta,n}+9$
$\kappa=9$	$J_{9,\theta,n+3}=8\cdot J_{9,\theta,n}\pm 1$, $J=Jp$	$m(p)_{9,\theta,n+6}=64m(p)_{9,\theta,n}\pm 7$
$\kappa=11$	$J_{11,\theta,n+5}=32\cdot J_{11,\theta,n}\pm 3$, $J=Jm(p)$	$m(p)_{11,\theta,n+10}=1024m(p)_{11,\theta,n}\pm 93$
$\kappa=13$	$J_{13,\theta,n+6}=64\cdot J_{13,\theta,n}\pm 5$, $J=Jm\cup Jp$	$m(p)_{13,\theta,n+12}=4096m(p)_{13,\theta,n}\pm 315$
$\kappa=15$	$J_{15,\theta,n+4}=16\cdot J_{15,\theta,n}+1$, $J=Jp$	$m(p)_{15,\theta,n+4}=256m(p)_{15,\theta,n}+17$
$\kappa=17$	$J_{17,\theta,n+4}=16\cdot J_{17,\theta,n}\pm 1$, $J=Jm\cup Jp$	$m(p)_{17,\theta,n+8}=256m(p)_{17,\theta,n}\pm 15$
$\kappa=19$	$J_{19,\theta,n+9}=512\cdot J_{19,\theta,n}\pm 27$, $J=Jm(p)$	$m(p)_{19,\theta,n+18}=262144m(p)_{19,\theta,n}\pm 13797$

Numbers (5) represent a superposition

$$\{m_{\kappa,\delta\theta,2n(2n+1)}\}_{n=0}^{n=\infty}\cup\{p_{\kappa,\theta,(2n+1)(2n)}\}_{n=0}^{n=\infty}=\{J^{\pm}_{\kappa,\theta,n}\}_{n=0}^{n=\infty} \tag{6}$$

of numbers

$$m_{\kappa,\theta,2n((2n+1)}=\frac{1}{\kappa}\left[\theta\cdot 2^{2n(2n+1)}-1\right] \tag{7}$$

and

$$m_{\kappa,\theta,2n((2n+1)}=\frac{1}{\kappa}\left[\theta\cdot 2^{2n(2n+1)}-1\right] \tag{8}$$

Exactly (6)-(8) are the Jacobsthal numbers that correctly form the branching (merging) nodes of the sequences $\{\theta\cdot 2^n\}_{n=0}^{n=\infty}$ [16]. For the root sequence, their values are given in Table 3. We see that in the model (6)-(8), the sequences CS_q of transformation $7q-1$ are isolated from the sequence $\{1\cdot 2^n\}_{n=0}^{n=\infty}$.

Table 3.

Numbers $m(p)_{\kappa,1,n}$ in the interval $\kappa=1\div 9$

$p_{9,1,n}$	$p_{7,1,n}$	$p_{5,1,n}$	$p_{3,1,n}$	$p_{1,1,n}$	2^n	$m_{1,1,n}$	$m_{3,1,n}$	$m_{5,1,n}$	$m_{7,1,n}$	$m_{9,1,n}$
	-	52429		65537	2^{16}	65535	21845	13107		
3641	-		10923	32769	2^{15}	32767			4681	
	-		m	16385	2^{14}	16383	5461			
	-	3277	2731	8193	2^{13}	8191				
	-			4097	2^{12}	4095	1365	819	585	455
	-		683	2049	2^{11}	2045				
	-	205		*1025*	$\mathbf{2^{10}}$	1023	341			
57	-		171	513	2^9	511			73	

	-			*257*	$\mathbf{2^8}$	255	85	51		
	--		43	129	2^7	127				
	-	13		***65***	$\mathbf{2^6}$	63	21		9	7
	-		11	**33**	2^5	31				
	-			*17*	$\mathbf{2^4}$	15	5	3		
1	-		3	9	2^3	7			1	
	-	1		*5*	$\mathbf{2^2}$	3	**1**			
	-		1	3	2^1	1				
	-			*2*	$\mathbf{2^0}$	0	0	0	**0**	0

The formulas for the numbers $m(p)_{\kappa,\vartheta,n}$ are given in the second column of Table 2. These relationships can be generalized in the form of a linear recurrent relation.

$$m(p)_{\kappa,1,n+T_m} = a \cdot m(p)_{\kappa,1,n} + b, \tag{9}$$

where for the initial parameter values $\kappa = 1 \div 17$, the periods T_n by powers are calculated in Table 4.

Table 4.

Period T_n

κ	1	3	5	7	9	11	13	15	17
T_n	1	2	4	6	6	10	12	4	8

In the reverse $n \to \infty$ direction, the number transformation algorithm is as follows:

$$m(p)_{\kappa,\theta,2n(2n\pm1)} = \frac{\theta \cdot 2^{2n(2n\pm1)} \mp 1}{\kappa} \Rightarrow \begin{cases} for \quad n \to \infty: \quad \dfrac{\theta \cdot 2^{2n(2n\pm1)} \mp 1}{\kappa} = odd, \\ for\, n \to 0: \quad \kappa \cdot m(p)_{\kappa,\delta,0} \pm 1 = \theta \cdot 2^{2n(2n\pm1)}, \end{cases} \quad \delta \in \mathrm{N}_{odd}, \tag{10}$$

by which odd numbers are formed in (9). However, it should be noted that transformations of type (8) and (9) or (10) are correct only in one direction of the degree change [17].

For the root $\{1 \cdot 2^n\}_{n=0}^{n=\infty}$ sequence, the following equality holds:

$$tst = n. \tag{11}$$

Therefore, let us consider the statistical patterns of number sequence formation for each iteration N of the bifurcation diagram of sequences $\{\theta \cdot 2^n\}_{n=0}^{n=\infty}$ of $tst \to \infty$ direction for some models (3). If $\kappa = 1$, then the Collatz function is written as $C_1 = 1 \cdot q \pm 1$. In this case, Jacobsthal numbers are calculated using the formula:

$$J_{\theta,n}^{\pm} = \theta \cdot 2^n \pm (-1)^n \quad \Rightarrow \quad \begin{cases} m_{\theta,2n(2n+1)} = \theta \cdot 2^{2n(2n+1)} - 1, \\ p_{\theta,(2n+1)(2n)} = \theta \cdot 2^{(2n+1)(2n)} + 1, \end{cases}, \tag{12}$$

where numbers of the type $M_n = 2^n - 1$, where $n \in N$ are known as Mersenne numbers, the first of which are: 1, 3, 7, 15, 31, 63, 127, 255, …, and numbers of the type $F_n = 2^{2^n} + 1$ are known as Fermat numbers. The first few Fermat numbers: 3, 5, 17, 257, 65537, 4294967297, 18446744073709551617, ... form a sequence known as A000215 in the OEIS classification [22].

For the numbers $m(p)_{1,\vartheta,n}$, the following equalities hold:

$$\theta \cdot 2^{n(2n\pm1)} = 1 \cdot m(p)_{1\text{б}\theta,n} + (-)1\,, \tag{13}$$

therefore, the periodicity by the degree n is equal to $T_n = 2^0 = 1$, and the numbers $m(p)_{1,\vartheta,n}$ satisfy the transformations .

$$\begin{array}{l} 2^0+0(2)=\mathbf{1(3)} \\ 2^1+1(3)=\mathbf{3(5)} \\ 2^2+3(5)=7(9) \\ 2^3+7(9)=\mathbf{15(17)} \\ 2^4+15(17)=\mathbf{31(33).} \end{array} \tag{14}$$

For the transformation (13), the tree of graphs in model (11) is constructed in Figure 2 for both functions $1q \pm 1$. Here, the starting number (unit) is doubled until the corresponding branching condition is met:

$$\theta \cdot 2^n \mp 1 = m(p)_{1,\theta,n}\,. \tag{15}$$

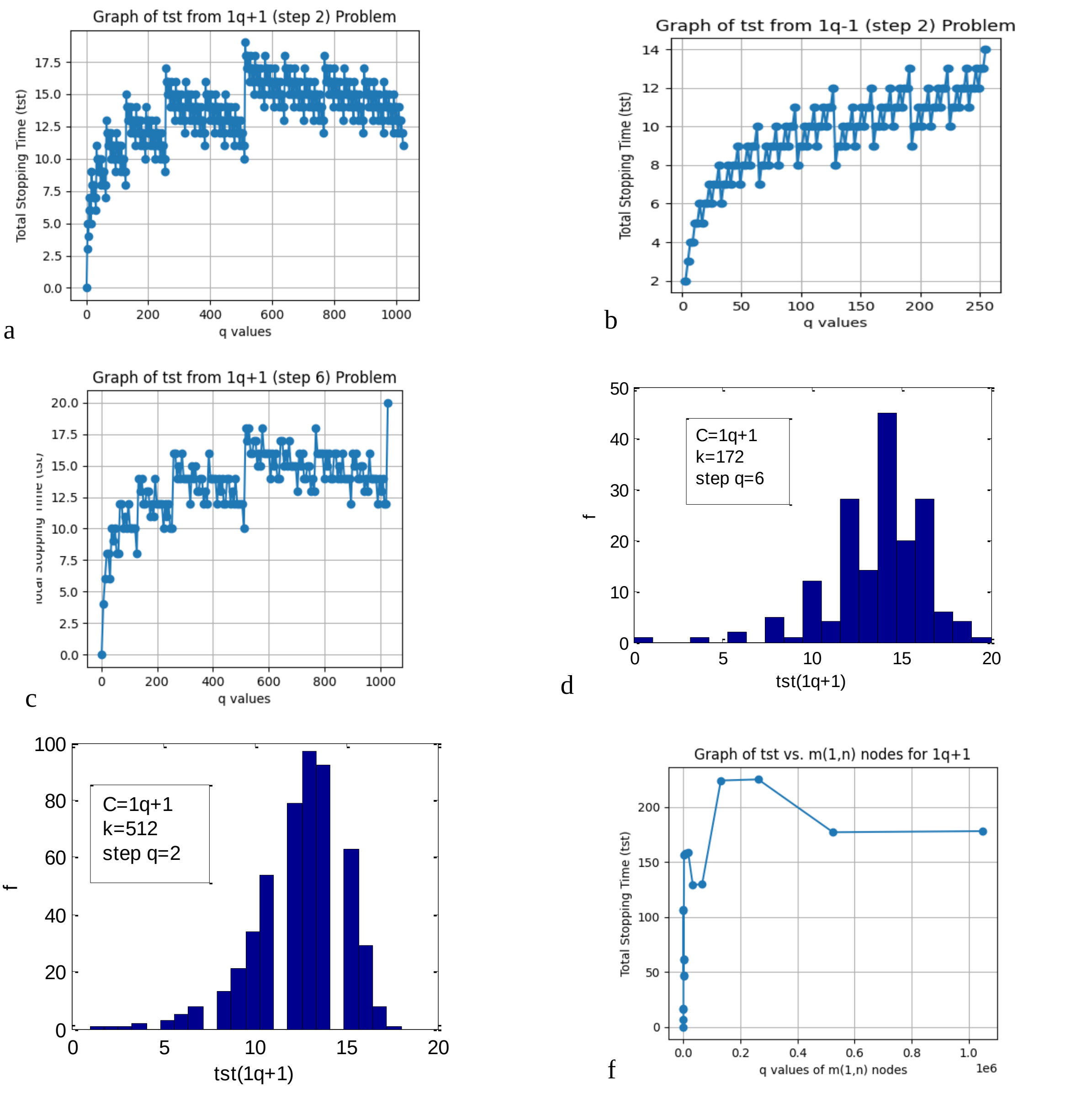
				1q-1								**1q+1**		
30														
29														**128**
56													**64**	**63**
23												**32**	**31**	
11											**16**			**62**
42	**15**									**8**		**15**		
41	**28**	**14**							**4**		**7**		**30**	**60**
80	**22**	**13**	**7**					**2**		**3**		**14**		**29**
38	**21**	**24**		**6**			**1**		**1**		**6**		**28**	
37	**40**	**11**	**12**		**3**	**2**						**12**	**13**	**56**
72	**19**	**20**	**10**	**5**	**4**							**5**		**27**
35	**36**	**18**	**9**	**8**									**24**	**26**
68	**34**	**17**	**16**										**11**	**48**
66	**33**	**32**											**10**	**23**
65	**64**													**22**
128														**20**
7	**6**	**5**	**4**	**3**	**2**	**1**	**tst**	**1**	**2**	**3**	**4**	**5**	**6**	**7**

Fig. 2. Branching tree in the direction $tst\ (n) \to \infty$

Here, the root sequence $\{1 \cdot 2^n\}_{n=0}^{n=\infty}$ is highlighted in yellow and in the columns, sequences of numbers with the same time tst are formed:

$$\cdot\left(\begin{array}{c} 1q+1 \\ {}_1\{2\}, \\ {}_2\{4\}, \\ {}_3\{3,8\}, \\ {}_4\{6,7,16\}, \\ {}_5\{5,12,14,15,32\}, \\ {}_6\{10,11,13,24,28,30,31,64\},\ldots \end{array}\right) \ \text{and} \ \left(\begin{array}{c} 1q-1 \\ {}_1\{2\}, \\ {}_2\{3,4\}, \\ {}_3\{5,61\}, \\ {}_4\{7,9,10,12,16\}, \\ {}_5\{11,13,14,17,18,20,24,32\}, \\ {}_6\{15,19,21,22,25,26,28,31,34,36,40,48,64\},\ldots \end{array}\right). \tag{16}$$

Thus, as a limitation on growth tst is absent, then with growth $tst(n) \to \infty$ the number k grows exponentially indefinitely individually for each column with an individual value tst (the index is written at the bottom left ${}_{tst}\{q\}$). Collatz sequences CS_{tst} (highlighted in green). In other words, each time value tst corresponds to an individual volume of numbers in sequences CS_{tst} of the same length.



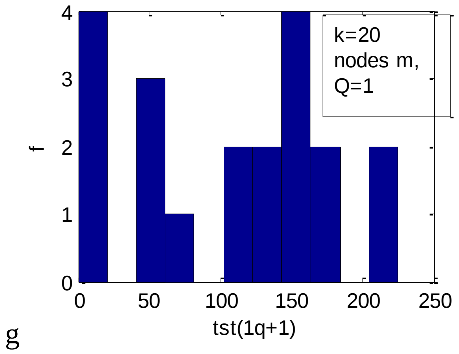


Fig. 3. Basic statistical regularities: a,b – graphs for $step = 2$; c – graph for $step = 6$; d – histogram for step 6; e – histogram for step 2; f,g – graph and histogram for time tst dependencies.

Let's consider a statistical model of dependency formation $tst(q)$, $k(q)$ based on the transformed number q, from the perspective of the branching model (10). The basic statistical patterns are depicted in the diagrams in Figure 3. Here, the interval $q = 1 \div 1023$ for the task $1q+1$ corresponds to the range of Jacobsthal numbers $m_{1,n}$, which form active nodes on the root sequence. Since there is a peculiar block structure of the spectrum, the studies were conducted for $step = 2$ (Figure 3a) and $step = 6$ (Figure 3d). Comparison with the corresponding histograms (Figure 3c, Figure 3f) indicates that in the interval $q = 1 \div 1023$, the most probable Collatz sequences have a length of tst =15. Similar statistics in Figures 3(e, g) for Jacobsthal numbers, indicate that the histograms appearance depends on the volume of data and the rules of their formation. As the data volume increases, the distribution curve (Figure 4a) in accordance with the central limit theorem, as the envelope of the histogram, approaches a normal distribution. When the task $1q+1$ changes to $1q-1$, the slope of the blocks (Figure 3a, b) changes mirror-wise; however, the statistical patterns tst remain similar. The histogram of the number distribution in column tst =9 for the transformation $1q+1$ is shown in Figure 4b.

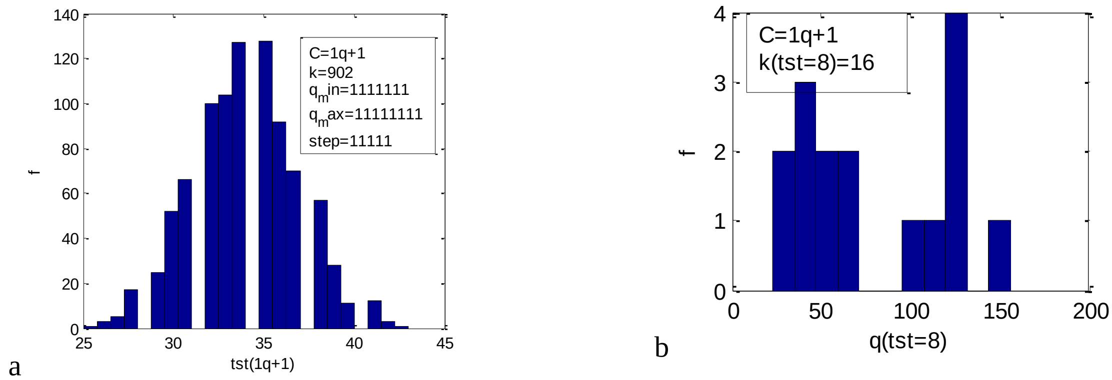


Fig. 4. $(1 \cdot q+1)$ histogram of the tst (a) and q

Now let's analyze the transformation model $3 \cdot q \pm 1\,(\kappa = 3)$. As shown in studies [16-19], in this model of natural number transformation, the recurrent integers of the nodes $m(p)_{3,\theta,n}$ are formed by division by 3.

$$m_{3,\theta,2n(2n+1)} = \frac{\theta \cdot 2^{2n(2n+)} - 1}{3},$$

$$p_{3,\theta,(2n+1)(2n)} = \frac{\theta \cdot 2^{(2n+1)((2n)} + 1}{3}, \qquad (17)$$

which are related to each other by the following relationship:

$$2m_{3,\theta,(2n+1)(2n)} = p_{3,\theta,2n(2m+1)} - 1, \qquad (18)$$

or

$$m(p)_{3,\theta,n+1} = 4m(p)_{3,\theta,n} \pm 1. \qquad (19)$$

Through the initial values $m(p)_{3,\theta,0(1)}$, $m(p)_{3,\theta,1(0)}$, the numbers $m(p)_{\theta,n}$ are expressed as:

$$m_{\theta,n} = 4^{\frac{n}{2}\left(\frac{n-1}{2}\right)} m_{\theta,0(1)} + \sum_{j=0}^{\frac{n-2}{2}\left(\frac{n-3}{2}\right)} 4^j, \quad p_{\theta,n} = 4^{\frac{n-1}{2}\left(\frac{n}{2}\right)} p_{\theta,1(0)} - \sum_{j=0}^{\left(\frac{n-3}{2}\right)\left(\frac{n-2}{2}\right)} 4^j, \qquad (20)$$

where $m(p)_{3,\theta,0(1)}$ are calculated under the condition:

$$If\quad \frac{\theta \pm 1}{3} = Integer \quad then \quad J^{\pm}_{\theta,0} = \frac{\theta \pm 1}{3} \quad and \quad J^{\pm}_{\theta,1} = \theta - J^{\pm}_{\theta,0}, \tag{21}$$

in accordance with the recurrence relation [16]:

$$J_{\theta,n+2} = J_{\theta,n+1} + 2J_{\theta,n}. \tag{22}$$

If the parameter is a multiple of three

$$\theta = \theta_3 = \text{int}\,eger \cdot 3, \tag{23}$$

then values $m_{3,\theta,2n(2n\pm1)}$, $p_{3,\theta,(2n\pm1)2n}$ are fractional and the sequence $\theta_3 \cdot 2^n$ in the form of a Jacobstal tree does not branch, as is the case for $\theta = 3,9,15,21,...$

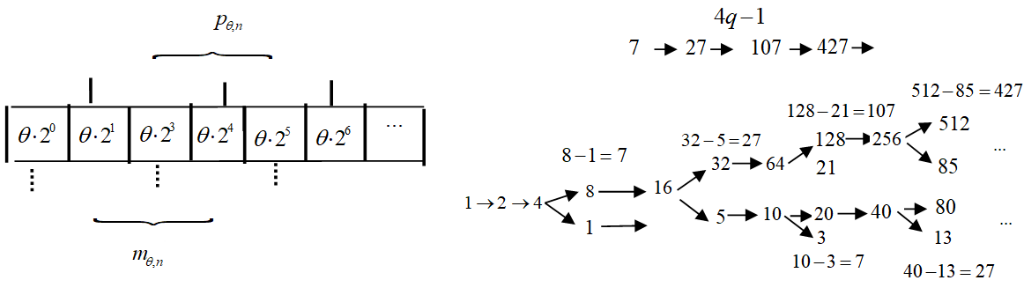


Fig.5. Nods in the points of $\{\vartheta \cdot 2^n\}_{n=0}^{n=\infty}$ sequence (a) and probability binary model (b)

Figure 5 illustrates the connection of Jacobstal numbers for the sequence $\{1 \cdot 2^n\}_{n=0}^{n=\infty}$ with Mersenne and Fermat numbers. Therefore, numbers of type $\theta \cdot 2^n - 1$ form Jacobstal numbers for nodes $m_{3,\theta,n}$, and numbers $\theta \cdot 2^n + 1$ form Jacobstal numbers for nodes $p_{3,\theta,n}$.

Table 5.

Connection of Jacobstal numbers for the sequence $\{1 \cdot 2^n\}_{n=0}^{n=\infty}$

2^tst-1	2^tst	2^tt+1	5·2^tst-1	5·2^tst	5·2^tt+1	7·2^tst-1	7·2^tst	7·2^tt+1
0/3=0	1	2	4	5	6/3=2	6/3=2	7	8
1	2	3/3=1	9/3=3	10	11	13	14	25/3=5
3/3=1	4	5	19	20	21/3=7	27/3=9	28	29
7	8	9/3=3	39/3=13	40	41	55	56	57/3=19
15/3=5	16	17	79	80	81/3=27	111/3=37	112	113
31	32	33/3=11	159/3=53	160	161	223	224	225/3=75
63/3=21	64	65	319	320	321/3=107	447/3=149	448	449

The principle of forming nodes with numbers $m(p)_{3,\theta,n}$ for sequence points $\{\theta \cdot 2^n\}_{n=0}^{n=\infty}$ is shown in Fig. 5a, the repetition period of which is equal $T_n = 4^1 = 4$ to the power n (Table 4). Therefore, in the direction $tst(n) \to \infty$ at the nodes with Jacobsthal numbers, the numbers are transformed according to the algorithm:

$$C^{\pm}(q) = \begin{cases} 2q, & q \equiv 0, \\ \dfrac{q \mp 1}{3}, & q \equiv 1 \end{cases}. \tag{24}$$

Fig. 5b illustrates the role of a sequence of inactive numbers. Let's build sequences $_{tst}\{q\}$ for the tree in Fig. 6a:

$$\cdot \begin{pmatrix} 3q+1 \\ {}_4\{2\}, \dots {}_4\{16\}, \\ {}_5\{5, 32\}, \\ {}_6\{10, 64\}, \\ {}_7\{3, 20, 21, 128\}, \\ {}_8\{6, 40, 42\, 256\}, \quad {}_4 \\ {}_9\{12, 13, 80, 84, 85, 512\}, \dots \end{pmatrix} \quad and \quad \begin{matrix} 3q-1 \\ {}_1\{2\}, \dots {}_3\{8\}, \\ {}_4\{3, 16\}, \\ {}_5\{6, 32\}, \\ {}_6\{12, 11, 64\}, \\ {}_7\{24, 22, 128\}, \\ {}_8\{48, 44, 43, \ 512\}, \dots \end{matrix} \quad . \tag{25}$$

A similar tree is built for transformation (Fig. 6b) with attractors 1.

tst														1				
1													2					k
2												4						1
3											8							1
4										16								1
5									32	5								2
6								64	10									2
7							128	21	20	3								4
8						256	42	6	40									4
9					512	85	84	12	80	13								4
10				1024	170	168	24	160		26								6
11			2048	341	340	336	320	48	53		52							8
12		4096	682	680	113	672	640	96		106	104	17						10
13	8192	1365	1364	1360	226	1344	1280	213	192	227	35	208	34	212				14
14	16384	2730	2728	2720	453	452	75	2688	2560	426	384	70	454	416	68	11	424	69	18

a

12			768	120	176	118	704	116	115	588	684	683	4006	
11				384	60	59	352	58		344	342	2048		
10				192	30		176	29	172	171	1024			
9					96	15	88		86	512				
8					48		44	43	256					
7					24		22	128						
6					12	11	64							
5					6	32								
4				3	16									
3				8										
2			4											
1		2												
Tst	1													

b

Fig. 6. Tree for: a) $3q+1$; b) $3q-1$

Let's consider the analysis of statistical charts. In Figure 7a, the histogram of the full stop time distribution tst for a relatively small range of numbers $q \in \mathrm{N}$ is presented. The graph in Figure 7b shows a flat two-dimensional distribution over a certain range tst of numbers. We can see that it consists of two maxima. As shown in Figure 7c, with the increase in the range of numbers q, the maxima blur and the envelope of the histogram $f_C(tst)$ takes on an asymmetric shape, resembling the well-known asymmetric Maxwell distribution $f_M(\vartheta)$ by velocities. For small argument values $f_M(\vartheta)$, the functions increase to the most probable value according to a parabolic law, while for larger argument values, they decrease exponentially. Parabolic growth and exponential decay are two mutually competing factors, whose compromise is achieved at the maximum of the distribution with the most probable argument value.

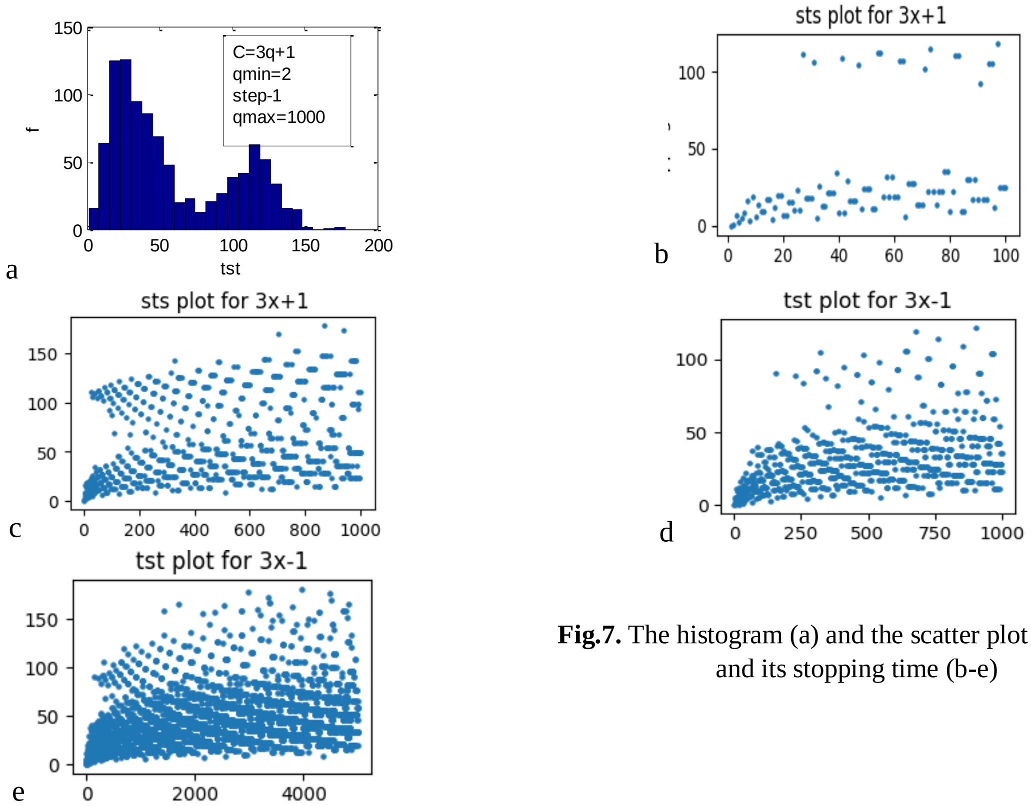


Fig.7. The histogram (a) and the scatter plot of numbers and its stopping time (b-e)

Essentially, the distribution function, as the envelope of the histogram, determines the frequency of realizations of the argument values f. Therefore, in the case of the Collatz problem, the envelope of the histogram in Figure 7a is a function of two arguments with different distributions:

$$f_C(tst,q) = f(q)\frac{dq}{d(tst)}, \qquad (26)$$

where $f(q)$ is the distribution function of the initial numbers q. Therefore, the time tst in the distribution (26) is a function of the number q. Thus, the shape of the envelope $tst = \varphi(q)$ will change with the variation in both the range of values q and its width. Now we will show that the spectra tst of the type shown in Figures 8a-e are determined by the patterns of the numbers $m(p)_{\kappa,\theta,n}$. For this purpose in Tables 6 and 7 calculate tst for the numbers $m(p)_{5,n}$ for the tasks $3q+1$ and $3q-1$.

Table 6.

Time tst ***for numbers*** $m(p)_{5,n}$ ***for the task*** $3q+1$

$m_{5,s}$	3	13	53	213	853	3413	13563	54613	218453	873813	3495253	13981013	55924053
tst	7	9	11	13	15	17	19	21	23	25	27	29	31
$p_{5,r}$	2	7	27	107	427	1707	6827	27307	109227	436907	1747627	6990507	27962027
tst		16	111	100	53	148	181	183	260	187	176	266	255

Table 7.

***Time** tst **for numbers** $m(p)_{5,n}$ **for the task** $3q-1$*

$m_{5,s}$	3	13	53	213	853	3413	13563	54613	218453	873813	3495253	13981013	55924053
tst	4	9	16	32	51	61	37	115	103	88	90	109	150
$p_{5,r}$	2	7	27	107	427	1707	6827	27307	109227	436907	1747627	6990507	27962027
tst		3	5	7	9	11	13	15	17	19	21	23	25

In the task $3q+1$ for nodes with numbers $m_{5,s}$, with the exponential increase of their values, the time tst increases by a constant value $\Delta = 2$ (Table 6). In the task $3q-1$, the nodes are formed by the numbers $p_{5,r}$, so as seen from Table 7, with the exponential increase of $p_{5,r}$, the time tst also increases by a constant value $\Delta = 2$A. Similar graph applies to inactive numbers m_{5_s} in the task $3q-1$. Therefore, the graphs $tst = \phi(q)$ in semi-logarithmic coordinates $\log\_q, tst$ will appear as straight lines [6,14] for active nodes and a scattered spectrum for inactive nodes.

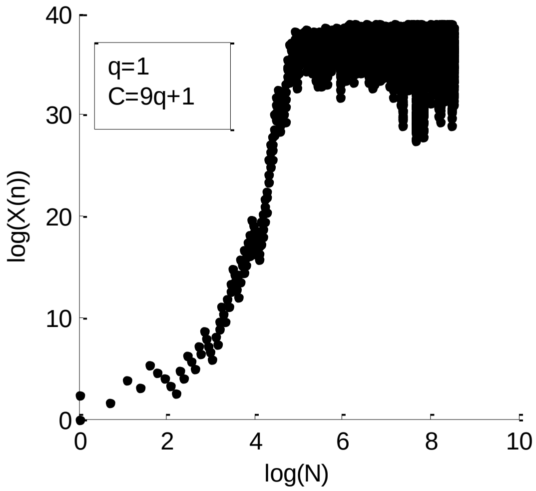


Fig.8. The CS_{9q+1} sequence, start q=1

As shown in [17], the transformation rules for numbers (2) and (12) are correct only in one direction of the degree n change. Therefore, we will demonstrate the possibility of constructing a fundamentally new model of a trivial cycle with a point attractor, which makes it possible to eliminate the unbounded growth of the transformation of unity, as is the case for transformation $C = 9q+1$:

$$\begin{aligned}&... \to 9\cdot 1+1=10 \to 5 \to 46 \to 23 \to 208 \to 104 \to 52 \to 26 \to 13 \to 118 \to 59 \to \\ &\to 532 \to 266 \to 133 \to 1198 \to 599 \to ...\end{aligned}. \qquad (27)$$

It is shown in Figure 8. Here $CS_{q=1}$ the sequence does not relax to a trivial periodic cycle with a point attractor, as in the case of the function $C = 3\cdot q \pm 1$ (2). This is because the function $C = \kappa \cdot q \pm 1$ is correct only in the direction $n \to 0$ (2), while in the reverse direction $n \to \infty$, the correct transformation is:

$$C(q) = \begin{cases} 2q, & q \equiv 0, \\ \dfrac{q-1}{3}, & q \equiv 1 \end{cases}. \qquad (28)$$

In the case of the transformation $C = 3\cdot q \pm 1$, the trajectories of unity by both approaches (2) and (28) formally coincide, unlike in the cases where κ >3. In the direction $n(tst) \to \infty$, trivial periodic cycles are formed as follows:

$$\begin{array}{l}
\kappa = 5: \\
1 \to 2 \to ... \to 16 \\
\qquad \downarrow \\
\qquad 5\cdot 3+1 = 16 \to ... \to 2 \to 1 \to 2 \to ... \to 16 \\
\qquad\qquad\qquad \downarrow \\
\qquad\qquad\qquad 5\cdot 3+1 = 16 \to ... \to 2 \to 1 \to ... \\
\kappa = 7: \\
1 \to 2 \to 4 \to 8 \\
\qquad \downarrow \\
\qquad 7\cdot 1+1 = 8 \to ... \to 2 \to 1 \to 2 \to 4 \to 8 \\
\qquad\qquad\qquad \downarrow \\
\qquad\qquad\qquad 7\cdot 1+1 = 8 \to 4 \to 2 \to 1 \to ... \\
\kappa = 9: \\
1 \to 2 \to ... \to 64 \\
\qquad \downarrow \\
\qquad 7\cdot 9+1 = 64 \to 32 \to ... \to 2 \to 1 \to 2 \to ... \to 64 \\
\qquad\qquad\qquad \downarrow \\
\qquad\qquad\qquad 7\cdot 9+1 = 64 \to 32 \to ... \to 2 \to 1 \to ...
\end{array} \tag{29}$$

.

where unbounded growth, as shown in Figure 8, is absent, whereas with the classical algorithm, the transformation of unity can grow without bound:

$$\begin{array}{ll}
\kappa = 3 & 1\cdot 3+1 = 4 \to 2 \to 1 \\
\kappa = 5 & 1\cdot 5+1 = 6 \to 3 \to 16 \to 8 \to 4 \to 2 \to 1 \\
\kappa = 7 & 1\cdot 7+1 = 8 \to 4 \to 2 \to 1 \\
\kappa = 9 & 1\cdot 5+1 = 6 \to 3 \to 16 \to 8 \to 4 \to 2 \to 1 \\
\kappa = 11 & 1\cdot 11+1 = 56 \to 28 \to 14 \to 7 \to 78 \to 39 \to 430 \to 215 \to 2366 \to 1183 \to 13014 \to 6507 \to ... \\
\kappa = 13 & 1\cdot 13+1 = 14 \to 7 \to 92 \to 46 \to 23 \to 300 \to 150 \to 75 \to 976 \to 488 \to 244 \to 122 \to 61 \to 794 \to 397 \to ... \\
\kappa = 15 & 1\cdot 15+1 = 16 \to 8 \to 4 \to 2 \to 1 \\
\kappa = 17 & 1\cdot 17+1 = 18 \to 9 \to 154 \to 77 \to 1310 \to 655 \to 11136 \to 5568 \to 2784 \to 1392 \to 696 \to 348 \to 174 \to 87 \to ... \\
\kappa = 19 & 1\cdot 19+1 = 20 \to 10 \to 5 \to 96 \to 48 \to 24 \to 12 \to 6 \to 3 \to 58 \to 29 \to 552 \to 276 \to 138 \to 69 \to ...
\end{array} . \tag{30}$$

Built on the basis of transformations (30) are shown in Fig. 9.

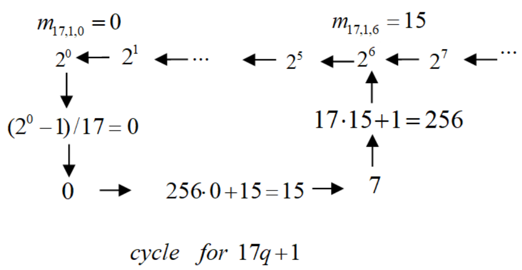


Fig. 9. The trivial cycle for function $C = 17q + 1$

Here, an arbitrary number q on the root sequence $\{1\cdot 2^n\}_{n=0}^{n=\infty}$, halving, reaches a value 2^0 and, according to rule (7), branches, forming a node $m_{17,1,0}=0$. Further, the Jacobsthal number $m_{17,1,0}=0$ according to the rule $m_{17,1,m+6}=256\cdot m_{9,1,m}+15$ transforms into the next Jacobsthal number $m_{17,1,6}=15$, which, being odd, transforms according to the rule $17q+1$ into the power number 2^6 of the sequence $\{1\cdot 2^n\}_{n=0}^{n=\infty}$. Such a trivial cycle, also periodic with a point singular attractor, allows the starting point of time tst to be aligned with the singular attractor $2^0=1$.

Conclusions

A new model of a branching tree in the direction of increasing power 2^n (merging in the reverse direction), which coincides with the direction of increasing total stop time, is proposed for the first time. It is shown that each time corresponds to a sequence of individual numbers, the volume of which increases as $n(tst)\to\infty$. Thus, it is proven that each time corresponds to a finite number of Collatz sequences of the same length. The reason for the formation of a histogram or spectrum $tst(q)$ with two maxima is established. It is shown that the double structure is formed by the regularities of the recurrent Jacobsthal numbers of the nodes of the sequences $\{\theta\cdot 2^n\}_{n=0}^{n=\infty}$. It is found that the graph $tst(q)$ with the numbers of active nodes in semi-logarithmic coordinates $tst,\log m(p)$ has the appearance of a straight line, while the graph for the numbers of inactive nodes has the appearance of a scattered spectrum. Based on the established statistical patterns $tst(q)$, a new recurrent model of trivial cycles is proposed.

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СТАТИСТИЧНЕ МОДЕЛЮВАННЯ СИСТЕМ ДИСКРЕТНОГО ПЕРЕТВОРЕННЯ ДАНИХ κ·q±1

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Анотація. Вперше запропонована нова модель дерева розгалужень в напрямку зростання степеня 2^n (злиття в реверсному напрямку), який співпадає з напрямком збільшення часу повної зупинки. Показано, що кожному часу відповідає послідовність індивідуальних чисел, обсяг якої зростає при $n(tst) \to \infty$. Таким чином доказано, що кожному часу відповідає скінчена кількість послідовностей Коллатца однакової довжини. Встановлена причина формування гістограми або спектру $tst(q)$ із двох максимумів. Показано, що подвійна структура формується закономірностями рекурентних чисел Якобсталя вузлів послідовностей $\{\theta \cdot 2^n\}_{n=0}^{n=\infty}$. Встановлено, що графік $tst(q)$ із числами активних вузлів в напівлогарифмічних координатах $tst, \log m(p)$ має вигляд прямої, тоді як графік для чисел неактивних вузлів, має вигляд розсіяного спектра. На основі встановлених статистичних закономірностей $tst(q)$, запропонована нова рекурентна модель тривіальних циклів.

Ключові слова: числа повторень, послідовності повторень, числа Якобсталя, гіпотеза Коллатца, загальний час зупинки, ймовірність, тривіальний цикл, послідовність Коллатца, гістограма, розсіяні спектри