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Numerical approximation of the MGT system with Fourier's law

Smouk A., Radid A.

Department of Mathematics and Informatics, Hassan II University, FSAC, Fundamental and Applied Mathematics Laboratory, Casablanca, Morocco

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In this paper, we consider the Moore–Gibson–Thompson–Fourier system made by coupling the Moore–Gibson–Thompson (MGT) equation with the classical Fourier heat equation known as the MGT–Fourier model. For $\sigma = \alpha\beta - \gamma > 0$, the authors used the semi-group method to prove the existence and uniqueness of global solutions and the exponential stability of total energy. Our contribution will consist in studying numerical method based on finite element discretization in the spacial variable x and finite difference schema in time of the MGT–Fourier model. A discrete stability property and a priori error estimates are proved. Finally, the numerical simulation agrees well with theoretical results.

Keywords: *MGT equation; Fourier's law; numerical stability; finite element method; numerical simulations.*

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1. Introduction

Historically, the earliest appearance of The Moore–Gibson–Thompson (MGT) equation was in a Stokes paper [1]. This equation was developed to represent wave propagation in viscous thermally relaxing fluids [2–5]. Then, they considered the following equation system:

$$u_{ttt} + \alpha u_{tt} - \beta \Delta u_t - \gamma \Delta u = 0. \tag{1}$$

Due to the substantial interest in the mathematical study of the MGT equation, there is a wide body of literature with various papers and references available [6–9]. Many recent works applied the classical model for heat propagation, which turns into the well-known equations for temperature θ and heat flux vector q

$$\theta_t + \varsigma \operatorname{div} q = 0 \tag{2}$$

and

$$q + \kappa \nabla \theta = 0 \tag{3}$$

with the constants ς and κ being positive. Replacing (3) (Fourier's law) into (2) results in the parabolic heat equation shown below

$$\theta_t - \varsigma\kappa \Delta \theta = 0. \tag{4}$$

In present paper, we consider the MGT equation using the Fourier's law given by coupling (1) and (4) in the following system

$$\begin{cases} u_{ttt} + \alpha u_{tt} - \beta \Delta u_t - \gamma \Delta u = -\eta \Delta \theta, \\ \theta_t - \kappa \Delta \theta = \eta \Delta u_{tt} + \alpha \eta \Delta u_t. \end{cases} \tag{5}$$

Where $x \in \Omega$, $t \in (0, \infty)$, and the function $u = u(x,t)$ represents the vibration of flexible structures, respectively, and $\theta = \theta(x,t)$ the temperature difference between the actual state and a reference temperature. The standard MGT parameters, α, β, γ, the thermal conductivity $\kappa > 0$, the coupling constant $\eta \neq 0$ and the domain $\Omega = [0, L]$. The initial conditions are given by

$$u(x,0) = u_0, \quad u_t(x,0) = u_1, \quad u_{tt}(x,0) = u_2, \quad \theta(x,0) = \theta_0, \tag{6}$$

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where $u_0, u_1, u_2, \theta_0 \colon \Omega \to \mathbb{R}$ are assigned initial data. The system is supplemented with the Dirichlet boundary conditions

$$u(0,t) = u(L,t) = \theta(0,t) = \theta(L,t) = 0. \tag{7}$$

Numerous studies [10, 11] of the MGT system with Fourier's law (5) show that the stability number

$$\sigma = \alpha\beta - \gamma$$

has a considerable impact on the MGT equation's stability features.

Now, we introduce new variables $y = u_t + \alpha u$, $v = u_t$ and using $\sigma = \alpha\beta - \gamma$, consequently the system (5) can be rewritten as

$$\begin{cases} y_{tt} - \dfrac{\gamma}{\alpha}\Delta y - \dfrac{\sigma}{\alpha}\Delta v + \eta\Delta\theta = 0, \\ \theta_t - \kappa\Delta\theta - \eta\Delta y_t = 0. \end{cases} \tag{8}$$

Then, the energy functional $\mathcal{E}(t)$ associated to (5)–(7) can be written as

$$\mathcal{E}(t) = \frac{1}{2}\left[\int_\Omega |y_t|^2 dx + \frac{\gamma}{\alpha}\int_\Omega |\nabla y|^2 dx + \frac{\sigma}{\alpha}\int_\Omega |\nabla v|^2 dx + \int_\Omega |\theta|^2 dx\right]. \tag{9}$$

The authors in [11] proved that the energy (9) decays exponentially for $\sigma > 0$ and verifies

$$\mathcal{E}'(t) = -\sigma\int_\Omega |\nabla v|^2 dx - \kappa\int_\Omega |\nabla\theta|^2 dx.$$

2. Numerical approximation

The system (5) with boundary conditions (7) and initial conditions (6) is approximated using finite elements in this section. On the basis of finite differences in time and finite elements in space, we present and investigate an implicit Euler type scheme. We establish the discrete energy decays.

Taking $w = z_t$; we rewrite system (8)

$$\begin{cases} w_t - \dfrac{\gamma}{\alpha}\Delta y - \dfrac{\sigma}{\alpha}\Delta v + \eta\Delta\theta = 0, \\ \theta_t - \kappa\Delta\theta - \eta\Delta w = 0. \end{cases} \tag{10}$$

To get the weak form associated to system (10), we multiply the equations by test functions $\zeta, \upsilon \in H^1(0,1)$ and integrating by parts,

$$\begin{cases} (w_t, \zeta) + \dfrac{\gamma}{\alpha}(\nabla y, \nabla\zeta) + \dfrac{\sigma}{\alpha}(\nabla v, \nabla\zeta) - \eta(\nabla\theta, \nabla\zeta) = 0, \\ (\theta_t, \upsilon) + \kappa(\nabla\theta, \nabla\upsilon) + \eta(\nabla w, \nabla\upsilon) = 0. \end{cases} \tag{11}$$

Consider a subdivision $0 = x_0 < x_1 < \ldots < x_{J-1} < x_J = 1$ of the domain $\Omega = [0,1]$ such that $x_j = jh$, $\forall j = 0, \ldots, J$ with $J > 0$ and the space step $h = 1/J$, we take

$$S^h = \left\{g \in H^1(0,1) | g \in C([0,L]), g|_{(x_j, x_{j+1})} \text{ is a linear polynomial, with } j = 0, \ldots, J-1\right\}$$

and

$$S_0^h = \left\{f \in S^h | f(0) = f(1) = 0\right\}.$$

Let the time step be $\Delta t = T/N$ for a final time T and a positive integer N and $t_n = n\Delta t$, $n = 0, \ldots, N$.

Using the backward Euler scheme, the finite element method for (11) is able to find, for $n = 1, \ldots, N$ and for all ζ_h, $\upsilon_h \in S^h$

$$\begin{cases} \dfrac{1}{\Delta t}(w_h^n - w_h^{n-1}, \zeta_h) + \dfrac{\gamma}{\alpha}(\nabla y_h^n, \nabla\zeta_h) + \dfrac{\sigma}{\alpha}(\nabla v_h^n, \nabla\zeta_h) - \eta(\nabla\theta_h^n, \nabla\zeta_h) = 0, \\ \dfrac{1}{\Delta t}(\theta_h^n - \theta_h^{n-1}, \upsilon_h) + \kappa(\nabla\theta_h^n, \nabla\upsilon_h) + \eta(\nabla w_h^n, \nabla\upsilon_h) = 0, \end{cases} \tag{12}$$

where

$$v_h^n = \frac{u_h^n - u_h^{n-1}}{\Delta t}, \quad y_h^n = v_h^n + \alpha u_h^n, \text{ and } w_h^n = \frac{y_h^n - y_h^{n-1}}{\Delta t}, \tag{13}$$

are approximations to $u_t(t_n)$, $v(t_n) + \alpha u(t_n)$, $y_t(t_n)$, respectively.

This inequality will be used frequently:

$$(a_1 - a_2, a_1) = \frac{1}{2}\left[\|a_1 - a_2\|^2 + \|a_1\|^2 - \|a_2\|^2\right]. \tag{14}$$

A discrete form of the energy decay property satisfied by the solution of system (5) is the next result.

Theorem 1. *Assume the discrete energy is:*

$$\mathcal{E}_h^n = \frac{1}{2}\left(\|w_h^n\|^2 + \frac{\gamma}{\alpha}\|\nabla y_h^n\|^2 + \frac{\sigma}{\alpha}\|\nabla v_h^n\|^2 + \|\theta_h^n\|^2\right).$$

Then, the decay property

$$\frac{\mathcal{E}_h^n - \mathcal{E}_h^{n-1}}{\Delta t} \leqslant 0,$$

holds for $n = 1, 2, \ldots, N$.

Proof. Taking $\zeta_h = w_h^n$ and $\upsilon_h = \theta_h^n$ in (12),

$$\begin{cases} \dfrac{1}{\Delta t}\left(w_h^n - w_h^{n-1}, w_h^n\right) + \dfrac{\gamma}{\alpha}\left(\nabla y_h^n, \nabla w_h^n\right) + \dfrac{\sigma}{\alpha}\left(\nabla v_h^n, \nabla w_h^n\right) - \eta\left(\nabla \theta_h^n, \nabla w_h^n\right) = 0, \\ \dfrac{1}{\Delta t}\left(\theta_h^n - \theta_h^{n-1}, \theta_h^n\right) + \kappa\left(\nabla \theta_h^n, \nabla \theta_h^n\right) + \eta\left(\nabla w_h^n, \nabla \theta_h^n\right) = 0. \end{cases} \tag{15}$$

Adding two equations of system (15), we have

$$\frac{\left(w_h^n - w_h^{n-1}, w_h^n\right)}{\Delta t} + \frac{\gamma}{\alpha}\left(\nabla y_h^n, \nabla w_h^n\right) + \frac{\sigma}{\alpha}\left(\nabla v_h^n, \nabla w_h^n\right) + \frac{\left(\theta_h^n - \theta_h^{n-1}, \theta_h^n\right)}{\Delta t} + \kappa\left(\nabla \theta_h^n, \nabla \theta_h^n\right) = 0.$$

Recalling (13) and (14),

$$\frac{1}{\Delta t}\left(w_h^n - w_h^{n-1}, w_h^n\right) = \frac{1}{2\Delta t}\left(\|w_h^n - w_h^{n-1}\|^2 + \|w_h^n\|^2 - \|w_h^{n-1}\|^2\right),$$

$$\begin{aligned} \frac{\gamma}{\alpha}\left(\nabla y_h^n, \nabla w_h^n\right) &= \frac{\gamma}{\alpha}\left(\nabla y_h^n, \frac{\nabla y_h^n - \nabla y_h^{n-1}}{\Delta t}\right) \\ &= \frac{\gamma}{2\alpha\Delta t}\left(\left\|\nabla y_h^n - \nabla y_h^{n-1}\right\|^2 + \|\nabla y_h^n\|^2 - \|\nabla y_h^{n-1}\|^2\right), \end{aligned}$$

$$\begin{aligned} \frac{\sigma}{\alpha}\left(\nabla v_h^n, \nabla w_h^n\right) &= \frac{\sigma}{\alpha}\left(\nabla v_h^n, \frac{\nabla y_h^n - \nabla y_h^{n-1}}{\Delta t}\right) \\ &= \frac{\sigma}{\alpha}\left(\nabla v_h^n, \frac{\Delta(v_h^n + \alpha u_h^n) - \Delta(v_h^{n-1} + \alpha u_h^{n-1})}{\Delta t}\right) \\ &= \frac{\sigma}{\alpha}\left(\nabla v_h^n, \frac{\nabla v_h^n - \nabla v_h^{n-1}}{\Delta t}\right) - \sigma\left(\nabla v_h^n, \frac{\Delta u_h^n - \Delta u_h^{n-1}}{\Delta t}\right) \\ &= \frac{\sigma}{\alpha}\left(\nabla v_h^n, \frac{\nabla v_h^n - \nabla v_h^{n-1}}{\Delta t}\right) + \sigma\left(\nabla v_h^n, \nabla v_h^n\right) \\ &= \frac{\sigma}{2\alpha\Delta t}\left(\|\nabla v_h^n - \nabla v_h^{n-1}\|^2 + \|\nabla v_h^n\|^2 - \|\nabla v_h^{n-1}\|^2\right) + \sigma\|\nabla v_h^n\|^2, \end{aligned}$$

$$\frac{1}{\Delta t}\left(\theta_h^n - \theta_h^{n-1}, \theta_h^n\right) = \frac{1}{2\Delta t}\left(\|\theta_h^n - \theta_h^{n-1}\|^2 + \|\theta_h^n\|^2 - \|\theta_h^{n-1}\|^2\right),$$

thus,

$$\begin{aligned} \frac{1}{2\Delta t}&\left(\|w_h^n - w_h^{n-1}\|^2 + \|w_h^n\|^2 - \|w_h^{n-1}\|^2\right) + \frac{\gamma}{2\alpha\Delta t}\left(\|\nabla y_h^n - \nabla y_h^{n-1}\|^2 + \|\nabla y_h^n\|^2 - \|\nabla y_h^{n-1}\|^2\right) \\ &+ \frac{\sigma}{2\alpha\Delta t}\left(\|\nabla v_h^n - \nabla v_h^{n-1}\|^2 + \|\nabla v_h^n\|^2 - \|\nabla v_h^{n-1}\|^2\right) + \sigma\|\nabla v_h^n\|^2 \\ &+ \frac{1}{2\Delta t}\left(\|\theta_h^n - \theta_h^{n-1}\|^2 + \|\theta_h^n\|^2 - \|\theta_h^{n-1}\|^2\right) + \kappa\|\nabla \theta_h^n\|^2 = 0. \end{aligned}$$

We conclude that

$$0 = \frac{1}{2\Delta t}\left(\|w_h^n - w_h^{n-1}\|^2 + \|w_h^n\|^2 - \|w_h^{n-1}\|^2\right) + \frac{\gamma}{2\alpha\Delta t}\left(\|\nabla y_h^n - \nabla y_h^{n-1}\|^2 + \|\nabla y_h^n\|^2 - \|\nabla y_h^{n-1}\|^2\right)$$
$$+ \frac{\sigma}{2\alpha\Delta t}\left(\|\nabla v_h^n - \nabla v_h^{n-1}\|^2 + \|\nabla v_h^n\|^2 - \|\nabla v_h^{n-1}\|^2\right) + \sigma\|\nabla v_h^n\|^2$$
$$+ \frac{1}{2\Delta t}\left(\|\theta_h^n - \theta_h^{n-1}\|^2 + \|\theta_h^n\|^2 - \|\theta_h^{n-1}\|^2\right) + \kappa\|\nabla\theta_h^n\|^2 \geqslant \frac{\mathcal{E}_h^n - \mathcal{E}_h^{n-1}}{\Delta t}.$$

So, $\frac{\mathcal{E}_h^n - \mathcal{E}_h^{n-1}}{\Delta t} \leqslant 0$ and the theorem is demonstrated using the notion of discrete energy. ∎

We now show the main error estimates result.

Theorem 2. *For any discretization parameters h and Δt, there exists a positive constant C independent from h and Δt such that for all $\{\zeta_h^i, v_h^i\}_N^{i=0} \subset S_0^h$,*

$$\max_{0\leqslant n\leqslant N}\left\{\|w^n - w_h^n\|^2 + \|\nabla y^n - \nabla y_h^n\|^2 + \|\nabla v^n - \nabla v_h^n\|^2 + \|\theta^n - \theta_h^n\|^2\right\}$$
$$\leqslant C\Delta t\sum_{i=1}^{N}\left(\|w_t^i - \delta w^i\|^2 + \|\nabla y_t^i - \delta\nabla y^i\|^2 + \|\nabla v_t^i - \delta\nabla v^i\|^2 + \|\theta_t^i - \delta\theta^i\|^2\right.$$
$$\left.+\|\nabla w^i - \nabla\zeta_h^i\|^2 + \|\nabla\theta^i - \nabla v_h^i\|^2\right) + C\max_{0\leqslant n\leqslant N}\left\{\|w^n - \zeta_h^n\|^2 + \|\theta^n - v_h^n\|^2\right\}$$
$$+ \frac{C}{\Delta t}\sum_{i=1}^{N-1}\left(\left\|w^i - \zeta_h^i - (w^{i+1} - \zeta_h^{i+1})\right\|^2 + \left\|\theta^i - v_h^i - (\theta^{i+1} - v_h^{i+1})\right\|^2\right)$$
$$+ C\left(\|w^0 - w_h^0\|^2 + \|\nabla y^0 - \nabla y_h^0\|^2 + \|\nabla v^0 - \nabla v_h^0\|^2 + \|\theta^0 - \theta_h^0\|^2\right),$$

where $\delta g^i = (g^i - g^{i-1})/\Delta t$.

Proof. First, we subtract the first variational equation in (11) at time $t = t_n$ for a test function $\zeta = \zeta_h \in S_0^h \subset S$ and the first discrete variational equation in (12) to obtain

$$(w_t^n - \delta w_h^n, \zeta_h) + \frac{\gamma}{\alpha}(\nabla y^n - \nabla y_h^n, \nabla\zeta_h) + \frac{\sigma}{\alpha}(\nabla v^n - \nabla v_h^n, \nabla\zeta_h)$$
$$- \eta(\nabla\theta^n - \nabla\theta_h^n, \nabla\zeta_h) = 0, \quad \forall\zeta_h \in S_0^h$$

and so, we have

$$(w_t^n - \delta w_h^n, w^n - w_h^n) + \frac{\gamma}{\alpha}\big(\nabla y^n - \nabla y_h^n, \nabla(w^n - w_h^n)\big) + \frac{\sigma}{\alpha}\big(\nabla v^n - \nabla v_h^n, \nabla(w^n - w_h^n)\big)$$
$$- \eta\big(\nabla\theta^n - \nabla\theta_h^n, \nabla(w^n - w_h^n)\big)$$
$$= (w_t^n - \delta w_h^n, w^n - \zeta_h) + \frac{\gamma}{\alpha}\big(\nabla y^n - \nabla y_h^n, \nabla(w^n - \zeta_h)\big) + \frac{\sigma}{\alpha}\big(\nabla v^n - \nabla v_h^n, \nabla(w^n - \zeta_h)\big)$$
$$- \eta\big(\nabla\theta^n - \nabla\theta_h^n, \nabla(w^n - \zeta_h)\big), \quad \forall\zeta_h \in S_0^h.$$

Taking into account that

$$\begin{aligned}
(w_t^n - \delta w_h^n, w^n - w_h^n) &= (w_t^n - \delta w^n, w^n - w_h^n) + (\delta w^n - \delta w_h^n, w^n - w_h^n)\\
&\geqslant (w_t^n - \delta w^n, w^n - w_h^n) + \frac{1}{2\Delta t}\left(\|w^n - w_h^n\|^2 - \|w^{n-1} - w_h^{n-1}\|^2\right),\\
\big(\nabla y^n - \nabla y_h^n, \nabla(w^n - w_h^n)\big) &= (\nabla y_t^n - \delta\nabla y_h^n, \nabla y^n - \nabla y_h^n)\\
&= (\nabla y_t^n - \delta\nabla y^n, \nabla y^n - \nabla y_h^n) + (\delta\nabla y^n - \delta\nabla y_h^n, \nabla y^n - \nabla y_h^n)\\
&\geqslant (\nabla y_t^n - \delta\nabla y^n, \nabla y^n - \nabla y_h^n)\\
&+ \frac{1}{2\Delta t}\left(\|\nabla y^n - \nabla y_h^n)\|^2 - \|\nabla y^{n-1} - \nabla y_h^{n-1}\|^2\right),\\
\big(\nabla v^n - \nabla v_h^n, \nabla(w^n - w_h^n)\big) &= \big(\nabla v^n - \nabla v_h^n, \nabla(y_t^n - \delta y_h^n)\big)\\
&= \big(\nabla v^n - \nabla v_h^n, \nabla((v_t^n + \alpha u_t^n) - (\delta v_h^n + \alpha\delta u_h^n))\big)\\
&= \big(\nabla v^n - \delta\nabla v_h^n, \nabla(v_t^n - \delta v_h^n)\big) + \alpha\big(\nabla v^n - \nabla v_h^n, \nabla(u_t^n - \delta u_h^n)\big)\\
&= \big(\nabla v^n - \delta\nabla v_h^n, \nabla(v_t^n - \delta v_h^n)\big) + \alpha\big(\nabla v^n - \nabla v_h^n, \nabla(v^n - v_h^n)\big)
\end{aligned} \tag{16}$$

$$
\begin{aligned}
&= \big(\nabla v^n - \delta\nabla v_h^n, \nabla(v_t^n - \delta v_h^n)\big) + \alpha\big\|\nabla v^n - \nabla v_h^n\big\|^2\\
&\geqslant \big(\nabla v_t^n - \delta\nabla v^n, \nabla v^n - \nabla v_h^n\big) + \alpha\,\|\nabla v^n - \nabla v_h^n\|^2\\
&\quad + \frac{1}{2\Delta t}\left(\|\nabla v^n - \nabla v_h^n\|^2 - \|\nabla v^{n-1} - \nabla v_h^{n-1}\|^2\right).
\end{aligned}
$$

Secondly, we subtract the second variation equation in (11) at time $t = t_n$ for a test function $\upsilon = \upsilon_h \in S_0^h \subset S$ and the second discrete variation equation in (12) to obtain

$$
(\theta_t^n - \delta\theta_h^n, \upsilon_h) + \kappa\,(\nabla\theta^n - \nabla\theta_h^n, \nabla\upsilon_h) + \eta\,(\nabla w^n - \nabla w_h^n, \nabla\upsilon_h) = 0,
$$

and so, we have

$$
\begin{aligned}
&(\theta_t^n - \delta\theta_h^n, \theta^n - \theta_h^n) + \kappa\big(\nabla\theta^n - \nabla\theta_h^n, \nabla(\theta^n - \theta_h^n)\big) + \eta\big(\nabla w^n - \nabla w_h^n, \nabla(\theta^n - \theta_h^n)\big)\\
&\quad = (\theta_t^n - \delta\theta_h^n, \theta^n - \upsilon_h) + \kappa\big(\nabla\theta^n - \nabla\theta_h^n, \nabla(\theta^n - \upsilon_h)\big) + \eta\big(\nabla w^n - \nabla w_h^n, \nabla(\theta^n - \upsilon_h)\big).
\end{aligned}
$$

Taking into consideration

$$
\begin{aligned}
(\theta_t^n - \delta\theta_h^n, \theta^n - w_h^n) &= (\theta_t^n - \delta\theta^n, \theta^n - \theta_h^n) + (\delta\theta^n - \delta\theta_h^n, \theta^n - \theta_h^n)\\
&\geqslant (\theta_t^n - \delta\theta^n, \theta^n - \theta_h^n) + \frac{1}{2\Delta t}\left(\|\theta^n - \theta_h^n\|^2 - \|\theta^{n-1} - \theta_h^{n-1}\|^2\right).
\end{aligned}
$$

From (16) and using several times Cauchy's inequality (17)

$$
b_1 b_2 \leqslant \varepsilon b_1^2 + \frac{1}{4\varepsilon} b_2^2, \quad b_1, b_2, \varepsilon \in \mathbb{R}, \quad \varepsilon > 0, \tag{17}
$$

it follows that

$$
\begin{aligned}
&\frac{1}{2\Delta t}\left(\|w^n - w_h^n\|^2 - \|w^{n-1} - w_h^{n-1}\|^2\right) + \frac{\gamma}{2\alpha\Delta t}\big(\|\nabla y^n - \nabla y_h^n\|^2 - \|\nabla y^{n-1} - \nabla y_h^{n-1}\|^2\big)\\
&\quad + \frac{\sigma}{2\alpha\Delta t}\big(\|\nabla v^n - \nabla v_h^n\|^2 - \|\nabla v^{n-1} - \nabla v_h^{n-1}\|^2\big) + \sigma\|\nabla v^n - \nabla v_h^n\|^2 - \eta(\nabla\theta^n - \nabla\theta_h^n, \nabla w^n - \nabla w_h^n)\\
&\qquad \leqslant C\left(\|w_t^n - \delta w^n\|^2 + \|w^n - w_h^n\|^2 + \|\nabla y_t^n - \nabla y^n\|^2 + \|\nabla y^n - \nabla y_h^n)\|^2 + \|\nabla v_t^n - \nabla v^n\|^2\right.\\
&\qquad\quad \left. + \|\nabla v^n - \nabla v_h^n)\|^2 + \|\nabla\theta^n - \nabla\theta_h^n\|^2 + \|w^n - \zeta_h\|^2 + \|\nabla w^n - \nabla\zeta_h\|^2\right)\\
&\qquad\quad + (\delta w^n - \delta w_h^n, w^n - \zeta_h), \quad \forall\zeta_h \in S_0^h.
\end{aligned} \tag{18}
$$

Using a similar formula, we get the following estimates: for all $\upsilon_h \in S_0^h$,

$$
\begin{aligned}
&\frac{1}{2\Delta t}\left(\|\theta^n - \theta_h^n\|^2 - \|\theta^{n-1} - \theta_h^{n-1}\|^2\right) + \kappa\|\nabla\theta^n - \nabla\theta_h^n\|^2 + \eta(\nabla w^n - \nabla w_h^n, \nabla\theta^n - \nabla\theta_h^n)\\
&\quad \leqslant C\big(\|\theta_t^n - \delta\theta^n\|^2 + \|\theta^n - \theta_h^n\|^2 + \|\nabla\theta^n - \nabla\theta_h^n\|^2 + \|\nabla w^n - \nabla w_h^n\|^2 + \|\theta^n - \upsilon_h\|^2\\
&\qquad + \|\nabla\theta^n - \nabla\upsilon_h\|^2\big) + (\delta\theta^n - \delta\theta_h^n, \theta^n - \upsilon_h).
\end{aligned} \tag{19}
$$

Combining estimates (18) and (19) it follows that, for all $\zeta_h, \upsilon_h \in S_0^h$,

$$
\begin{aligned}
&\frac{1}{2\Delta t}\left(\|w^n - w_h^n\|^2 - \|w^{n-1} - w_h^{n-1}\|^2\right) + \frac{\gamma}{2\alpha\Delta t}\left(\|\nabla y^n - \nabla y_h^n\|^2 - \|\nabla y^{n-1} - \nabla y_h^{n-1}\|^2\right)\\
&\quad + \frac{\sigma}{2\alpha\Delta t}\left(\|\nabla v^n - \nabla v_h^n\|^2 - \|\nabla v^{n-1} - \nabla v_h^{n-1}\|^2\right) + \sigma\|\nabla v^n - \nabla v_h^n\|^2 - \eta(\nabla\theta^n - \nabla\theta_h^n, \nabla w^n - \nabla w_h^n)\\
&\quad \frac{1}{2\Delta t}\left(\|\theta^n - \theta_h^n\|^2 - \|\theta^{n-1} - \theta_h^{n-1}\|^2\right) + \kappa\|\nabla\theta^n - \nabla\theta_h^n\|^2 + \eta(\nabla w^n - \nabla w_h^n, \nabla\theta^n - \nabla\theta_h^n)\\
&\quad \leqslant C\big(\|w_t^n - \delta w^n\|^2 + \|w^n - w_h^n\|^2 + \|\nabla y_t^n - \nabla y^n\|^2 + \|\nabla y^n - \nabla y_h^n\|^2 + \|\nabla v_t^n - \nabla v^n\|^2\\
&\quad + \|\nabla v^n - \nabla v_h^n)\|^2 + \|\nabla\theta^n - \nabla\theta_h^n\|^2 + \|w^n - \zeta_h\|^2 + \|\nabla w^n - \nabla\zeta_h\|^2 + \|\theta_t^n - \delta\theta^n\|^2 + \|\theta^n - \theta_h^n\|^2\\
&\quad + \|\nabla\theta^n - \nabla\theta_h^n\|^2 + \|\nabla w^n - \nabla w_h^n\|^2 + \|\theta^n - \upsilon_h\|^2 + \|\nabla\theta^n - \nabla\upsilon_h\|^2\big)\\
&\quad + (\delta w^n - \delta w_h^n, w^n - \zeta_h) + (\delta\theta^n - \delta\theta_h^n, \theta^n - \upsilon_h).
\end{aligned}
$$

By multiplying the above estimations by Δt and adding up to n, we get, for all $\zeta_h, \upsilon_h \in S_0^h$,

$$
\begin{aligned}
&\|w^n - w_h^n\|^2 + \|\nabla y^n - \nabla y_h^n)\|^2 + \|\nabla v^n - \nabla v_h^n\|^2 + \|\theta^n - \theta_h^n\|^2\\
&\quad \leqslant C\Delta t\sum_{i=0}^{n}\left(\|w_t^i - \delta w^i\|^2 + \|w^i - w_h^i\|^2 + \|\nabla y_t^i - \nabla y^i\|^2 + \|\nabla y^i - \nabla y_h^i\|^2 + \|\nabla v_t^i - \nabla v^i\|^2\right.\\
&\qquad + \|\nabla v^i - \nabla v_h^i\|^2 + \|\nabla\theta^i - \nabla\theta_h^i\|^2 + \|\nabla w^i - \nabla\zeta_h^i\|^2 + \|\theta_t^i - \delta\theta^i\|^2 + \|\theta^i - \theta_h^i\|^2
\end{aligned}
$$

$$
\begin{aligned}
&+\|\nabla\theta^i-\nabla\theta_h^i\|^2+\|w^i-\zeta_h^i\|^2+\|\nabla w^i-\nabla w_h^i\|^2+\|\theta^i-\upsilon_h^i\|^2+\|\nabla\theta^i-\nabla\upsilon_h^i\|^2\big)\\
&\quad+\Delta t\sum_{i=0}^{n}\left((\delta w^i-\delta w_h^i,w^i-\zeta_h^i)+(\delta\theta^i-\delta\theta_h^i,\theta^i-\upsilon_h^i)\right)\\
&\qquad+C\left(\|w^0-w_h^0\|^2+\|\nabla y^0-\nabla y_h^0\|^2\|\nabla v^0-\nabla v_h^0\|^2+\|\theta^0-\theta_h^0\|^2\right).
\end{aligned}
$$

Finally, taking into consideration

$$
\begin{aligned}
\Delta t\sum_{i=1}^{n}\left(\delta w^i-\delta w_h^i,w^i-\zeta_h^i\right)&=(w^n-w_h^n,w^n-\zeta_h^n)+\left(w_h^0-y^1,w^1-\zeta_h^1\right)\\
&\quad+\sum_{i=1}^{n-1}\left(w^i-w_h^i,w^i-\zeta_h^i-\left(w^{i+1}-\zeta_h^{i+1}\right)\right),\\
\Delta t\sum_{i=1}^{n}\left(\delta\theta^i-\delta\theta_h^i,\theta^i-\upsilon_h^i\right)&=(\theta^n-\theta_h^n,\theta^n-\upsilon_h^n)+\left(\theta_h^0-\theta^0,\theta^1-\upsilon_h^1\right)\\
&\quad+\sum_{i=1}^{n-1}\left(\theta^i-\theta_h^i,\theta^i-\upsilon_h^i-\left(\theta^{i+1}-\upsilon_h^{i+1}\right)\right),
\end{aligned}
$$

so, we achieve adequate a priori error estimates by applying a discrete variant of Gronwall's inequality (see [12]). ■

The estimates in the previous theorem can be applied to determine the convergence order of the approximations provided by the discrete problem (12). As an example, we assume the regularity:

$$
\begin{aligned}
&u\in H^4\left(0,T;L^2(0,1)\right)\cap H^3\left(0,T;H^1(0,1)\right)\cap C^2\left([0,T];H^3(0,1)\right),\\
&\theta\in H^2\left(0,T;L^2(0,1)\right)\cap H^1\left(0,T;H^1(0,1)\right)\cap C\left([0,T];H^3(0,1)\right),
\end{aligned}
$$

we obtain the algorithm's linear convergence by using particular results on finite element approximation (see [13]) and earlier estimates derived in [12]. Here is our result.

Corollary 1. *Due to the assumptions of Theorem 2, there exists a positive constant $C>0$ that is independent of the discretization parameters h and Δt, such that*

$$
\max_{0\leqslant n\leqslant N}\left\{\|w^n-w_h^n\|^2+\|\nabla y^n-\nabla y_h^n\|^2+\|\nabla v^n-\nabla v_h^n\|^2+\|\theta^n-\theta_h^n\|^2\right\}\leqslant C(h+\Delta t).
$$

The numerical schemes were implemented using MATLAB on a Intel Core i5-6006U CPU @ 2.00 GHz.

3. Numerical simulation

Now, we give some numerical tests to validate the theoretical results.

3.1. Example 1: error estimate

The goal of the first example is to demonstrate the correctness and efficiency of the suggested fully discrete example. As a result, we will address this problem:

$$
\begin{aligned}
&w_t-\frac{\gamma}{\alpha}\Delta y-\frac{\sigma}{\alpha}\Delta v+\eta\Delta\theta=f \text{ in } (0,1)\times(0,T),\\
&\theta_t-\kappa\Delta\theta-\eta\Delta w=g \text{ in } (0,1)\times(0,T),\\
&u(x,0)=u_0(x),\quad u_t(x,0)=u_1(x),\quad \text{for a.e.}\quad x\in(0,1),\\
&u_{tt}(x,0)=u_2(x),\quad \theta(x,0)=\theta_0(x)\quad \text{for a.e.}\quad x\in(0,1),\\
&u(0,t)=u(1,t)=\theta(0,t)=\theta(1,t)=0\quad \text{for a.e.}\quad t\in(0,T),
\end{aligned}
\tag{20}
$$

with the following data:

$$
T=1,\quad \alpha=10^{-1},\quad \beta=1,\quad \gamma=10^{-2},\quad \eta=10^{-2},\quad \kappa=10^{-2}. \tag{21}
$$

In (20), the supply terms f and g are given by the following expressions, for all $(x,t)\in(0,1)\times(0,T)$,

$$
\begin{aligned}
&f(x,t)=e^t\left(2\beta+2\gamma+(1+\alpha)x-(1+\alpha)x^2+2\eta\,\pi\cos(\pi x)-\eta\,x\pi^2\sin(\pi x)\right),\\
&g(x,t)=e^{t^2}\left(2\eta+2\alpha\eta+x\sin(\pi x)-2\kappa\pi\cos(\pi x)+\kappa x\pi^2\sin(\pi x)\right).
\end{aligned}
\tag{22}
$$

Obviously, the analysis described in the preceding part may be easily applied to this slightly changed case. For all $x \in (0,1)$, the initial conditions are defined by

$$u_0(x) = u_1(x) = u_2(x) = x(1-x),$$
$$\theta_0(x) = x\sin(\pi x).$$

The exact solution to (20) can be easily obtained by

$$u(x,t) = e^t x(x-1), \quad \theta(x,t) = e^t x\sin(\pi x), \quad \forall (x,t) \in [0,1]\times[0,T].$$

The discretized solutions of (21) using implicit Euler type scheme based on finite differences in time step n and finite elements in space and taking account of $v = u_t$, $y = v + \alpha u$ and $w = y_t$ which are given by the following system

$$\begin{cases} U^n - \Delta t\, V^n = U^{n-1}, \\ V^n + \alpha\, U^n - Y^n = O^n, \\ Y^n - \Delta t\, W^n = Y^{n-1}, \\ MW^n + \dfrac{\gamma}{\alpha}\Delta t\, RY^n + \dfrac{\sigma}{\alpha} R\,\Delta t\, V^n - \eta\,\Delta t\, R\Theta^n = MW^{n-1} + F^n, \\ (M + \kappa\,\Delta t\, R)\,\Theta^n + \eta\,\Delta t\, RW^n = M\Theta^{n-1} + G^n, \end{cases}$$

where the vectors $U^n = (u_i^n)_{0\leqslant i\leqslant J}$, $V^n = (v_i^n)_{0\leqslant i\leqslant J}$, $W^n = (w_i^n)_{0\leqslant i\leqslant J}$, $Y^n = (y_i^n)_{0\leqslant i\leqslant J}$ and $\Theta^n = (\theta_i^n)_{0\leqslant i\leqslant J}$, for the matrices M and R are the mass matrix and the stiffness matrix respectively and F^n and G^n are supply terms.

Consequently, the numerical errors provided by

$$\max_{0\leqslant n\leqslant N} \left\{\|w^n - w_h^n\|^2 + \|\nabla y^n - \nabla y_h^n\|^2 + \|\nabla v^n - \nabla v_h^n\|^2 + \|\theta^n - \theta_h^n\|^2\right\}$$

are displayed in Table 1 for some discretization parameters values. Furthermore, we plot errors in Figure 1 and logarithmic errors in Figure 2 based on the specified parameter $h + \Delta t$. We conclude that the convergence is linear as demonstrated in the previous section for a given regularity of the continuous solution.

Table 1. Errors for $T = 1$.

J	Δt	Error
25	0.04	1.3323
50	0.02	0.6615
100	0.01	0.3306
200	0.005	0.1663
400	0.0025	0.0845

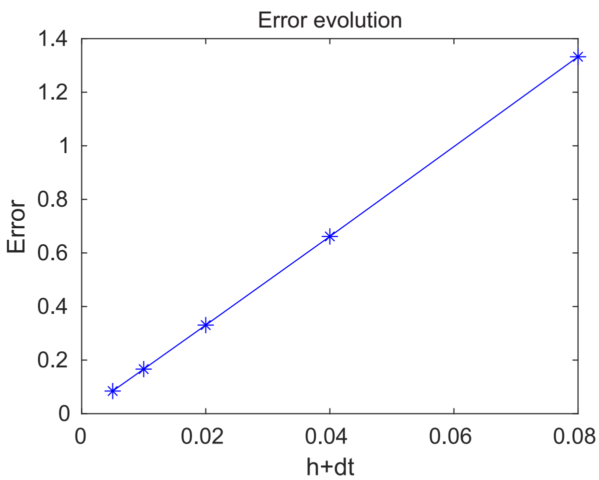

Fig. 1. Asymptotic behavior of Error.

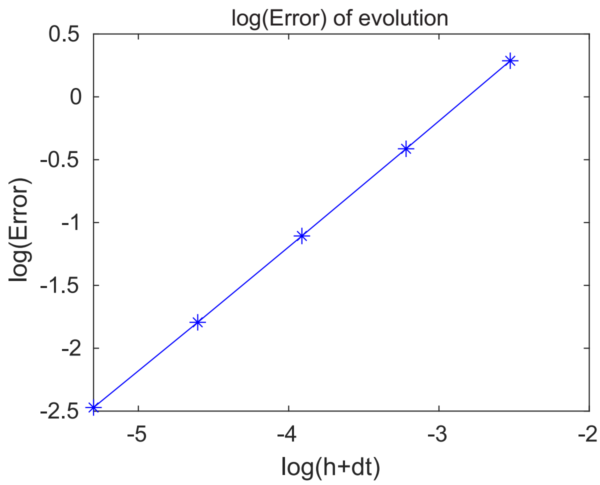

Fig. 2. Asymptotic behavior of log(Error).

3.2. Example 2: discrete energy

Assume that the supply terms have vanished and that the final time is $T = 80$. We also use the following data: $\alpha = 1$, $\beta = 3$, $\gamma = 2$, $\eta = 4$, $\kappa = 1$, and the following initial conditions, for all $x \in (0,1)$,

$$u_0(x) = u_1(x) = u_2(x) = \sin(\pi x)^2,$$
$$\theta_0(x) = \cos\left(\frac{\pi x}{2}\right)^2.$$

Taking the parameters $h = 2\Delta t = 0.0025$ and the discrete energy:

$$\mathcal{E}_h^n = \frac{1}{2}\left(\|w_h^n\|^2 + \frac{\gamma}{\alpha}\|\nabla y_h^n\|^2 + \frac{\sigma}{\alpha}\|\nabla v_h^n\|^2 + \|\theta_h^n\|^2\right).$$

Figures 3 and 4 show the evolution of discrete energy and discrete logarithmic energy. We can easily see that the energy decays exponentially.

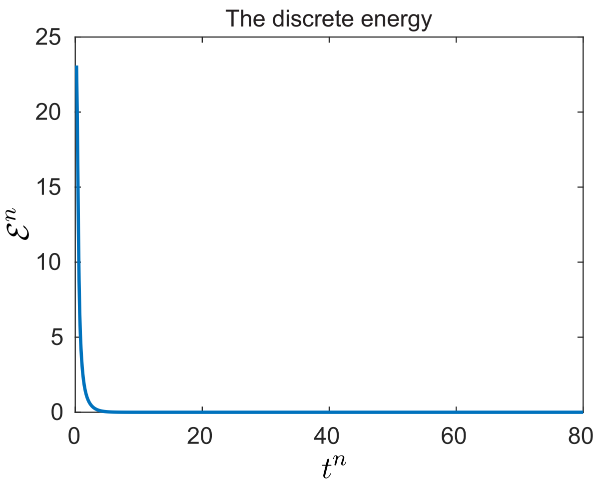

Fig. 3. Natural scale behavior of $\mathcal{E}_h^n$.

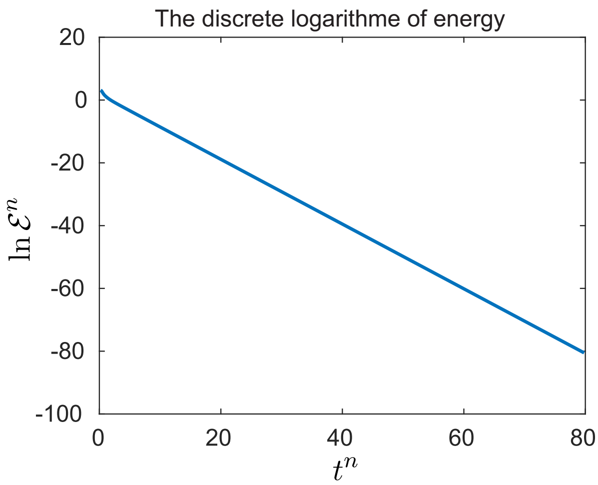

Fig. 4. Semi-log scale behavior of $\mathcal{E}_h^n$.

3.3. Example 3: the exact solution

In this last example, we will plot the exact solutions of problems (1)–(3). Again, we suppose (22) and the data: $T = 1$, $\alpha = 10^{-1}$, $\beta = 10^{-1}$, $\gamma = 10^{-3}$, $\eta = 10^{-2}$, $\kappa = 10^{-2}$.

If the initial conditions are the next

$$u_0(x) = u_1(x) = u_2(x) = x(x-1), \quad \forall x \in (0,1),$$
$$\theta_0(x) = x\sin(\pi x), \quad \forall x \in (0,1),$$

using the values $h = 0.005$ and $\Delta t = 0.0025$ the solution to discrete problem (12) is plotted in Figures 5 and 6. From these pictures, we can conclude that the solutions decay to zero.

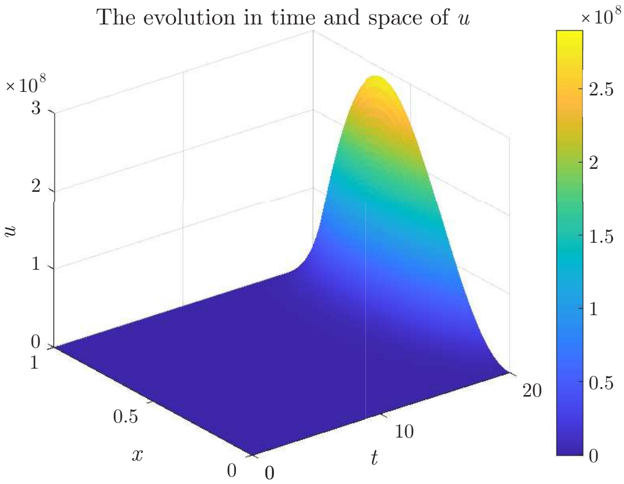

Fig. 5. Evolution of u_h^n.

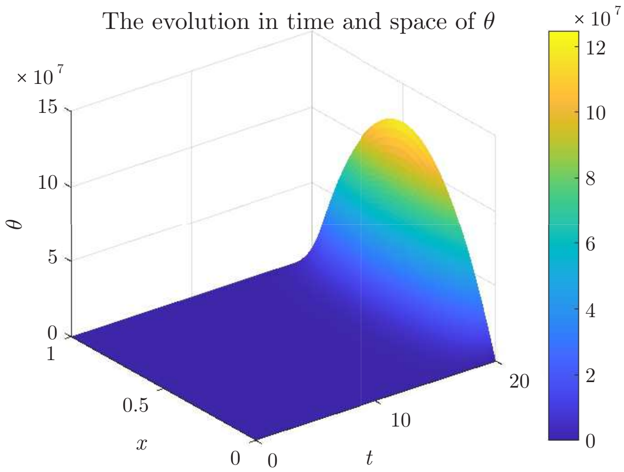

Fig. 6. Evolution of θ_h^n.

4. Conclusion

In this paper, we carried out the numerical study of the MGT system with the Laplace–Dirichlet operator $-\Delta$ taking into account Fourier's law in one dimension with Dirichlet boundary conditions.

Firstly, we introduced a numerical scheme based on finite element discretization P1 in space variable and the finite difference scheme in time which allows us to approach discrete energy. Also, we demonstrated the property of energy decay. Then, a priori error estimates for the semi-discrete and fully discrete schemes are established. Finally some numerical experiments were carried out for this system, the order of convergence of which agrees with that expected from the theories.

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Чисельна апроксимація системи МГТ зі законом Фур'є

Смук А., Радід А.

Факультет математики та інформатики Університету Хасана II,
FSAC, Лабораторія фундаментальної та прикладної математики, Касабланка, Марокко

У цій роботі розглядається система Мура–Гібсона–Томпсона–Фур'є, яка отримана об'єднанням рівняння Мура–Гібсона–Томпсона (MGT) з класичним рівнянням теплопровідності Фур'є, відома як модель MGT-Фур'є. Для $\sigma = \alpha\beta - \gamma > 0$ автори використали метод півгруп, щоб довести існування та єдиність глобальних розв'язків та експоненціальну стійкість повної енергії. Наш внесок полягає у вивченні чисельного методу, який заснований на скінченно-елементній дискретизації за просторовою змінною x та скінченно-різницевій схемі за часом моделі MGT–Фур'є. Доведено властивість дискретної стійкості та апріорні оцінки похибки. Накінець, числове моделювання добре узгоджується з теоретичними результатами.

Ключові слова: *рівняння MGT; закон Фур'є; чисельна стійкість; метод скінченних елементів; чисельне моделювання.*