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Dynamics of a diffusive business cycle model with two delays and variable depreciation rate

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The main aim of this work is to analyze the dynamics of a delayed business cycle model described by partial differential equations (PDEs) in order to take into account the depreciation rate of capital stock and the diffusion effect. Firstly, the existence of solutions and the economic equilibrium are carefully studied. Secondly, the local stability and the existence of Hopf bifurcation are established. Finally, some numerical simulations are presented to illustrate the analytical results.

Keywords: *business cycle; depreciation rate; diffusion effect; Hopf bifurcation; stability.*
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1. Introduction

Most mathematical models describing the dynamics business cycle like [1–4] assumed that the rate of depreciation of stock capital was constant. However, this rate is not constant, and there are few models that have investigated the dynamics of business cycle with variable depreciation rate. For instance, Mao and Liu [5] studied the dynamics of business cycle model by considering that the depreciation of capital as a decreasing function of capital stock. In 2023, Lasfar et al. [6] improved and generalized the model of Mao and Liu [5] by proposing the following nonlinear system

$$\begin{cases} \dfrac{dY}{dt} = \alpha[I(Y(t),K(t)) - \gamma Y(t)], \\ \dfrac{dK}{dt} = I(Y(t-\tau_1),K(t-\tau_2)) - \delta(K)K(t), \end{cases} \tag{1}$$

where $Y(t)$ represents the gross product, $K(t)$ refers to capital stock at time t. τ_1 is the delay between the investment decision and its execution. τ_2 is the time lag for the completion of investment. Further, $\gamma \in (0,1)$ is a coefficient called the saving constant. $I(Y,K)$ represents the investment function and it is assumed to be continuously differentiable in $\mathbb{R}^2$ with $\frac{\partial I}{\partial Y} > 0$ and $\frac{\partial I}{\partial K} < 0$. The parameter α denotes the adjustment coefficient in the goods market. Finally, the depreciation rate function is represented by $\delta(K)$.

On the other hand, the system (1) neglected the diffusion effects of economic activities and the regional differences. Economists have noticed that the general diffusion of the economy allows a diversified production, with high added value and strong complementary between the various economic sectors. In addition, the growth of economic activities is diffused by the increase in the rate of investment. So, it is important to study the dynamical behaviors of business cycle by taking into account the diffusion effects.

Motivated by above economical and mathematical considerations, we propose a mathematical model that takes into account both effects of variable depreciation rate and regional differences on the dynamics of business cycle. To do this, the present paper is organized as follows. Section 2 is devoted to the formulation of the model and presents some preliminary results. In Section 3, the existence and

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uniqueness of solution are proved. In Section 4, we investigate the local stability of the economic equilibrium and the existence of Hopf bifurcation. Some numerical simulations of our model are presented in Section 5. Finally, the paper ends with a conclusion and future research.

2. Preliminaries and model formulation

In this section, we present our economic cycle model with variable depreciation rate by taking into account the diffusion effect. This model is given by the following system of PDEs:

$$\begin{cases} \dfrac{\partial Y(t,x)}{\partial t} = d_1 \Delta Y(t,x) + \alpha[I(Y(t,x),K(t,x)) - \gamma Y(t,x)], \\ \dfrac{\partial K(t,x)}{\partial t} = d_2 \Delta K(t,x) + I(Y(t-\tau_1,x),K(t-\tau_2,x)) - \delta(K(t,x))K(t,x), \end{cases} \tag{2}$$

where $Y(t,x)$ and $K(t,x)$ are the gross product and capital stock at location x and time t. Δ is the Laplacian operator besides d_1 and d_2 are the diffusion coefficients of Y and K, respectively.

We consider model (2) with initial conditions:

$$Y(t,x) = \Phi_1(t,x), \quad K(t,x) = \Phi_2(t,x), \quad (t,x) \in [-\tau,0] \times \overline{\Omega}, \tag{3}$$

where $\tau = \max\{\tau_1,\tau_2\}$, and Neumann boundary conditions:

$$\frac{\partial Y}{\partial \xi} = \frac{\partial K}{\partial \xi} = 0 \text{ on } (0,+\infty) \times \partial\Omega, \tag{4}$$

where Ω is the market capacity and $\frac{\partial}{\partial \xi}$ indicates the outward normal derivative on the smooth boundary $\partial\Omega$.

After that, we give the necessary definitions and results that are needed for the proofs of the main results.

Lemma 1 (Ref. [7]). *Let A, B and D be three constants with $B \neq 0$. Consider the following problem*

$$\begin{cases} \dfrac{\partial u}{\partial t} - d_2 \Delta u \leqslant A - Bu, \quad x \in \Omega, \quad t > 0, \\ \dfrac{\partial u}{\partial \nu} = 0, x \in \partial\Omega, \quad t > 0, \\ u(0,x) = u_0(x), \quad x \in \bar{\Omega}. \end{cases}$$

Then $u(x,t) \leqslant \max_{x \in \overline{\Omega}} u_0(x)\, e^{-Bt} + \frac{A}{B}(1 - e^{-Bt})$. Moreover, if $B > 0$, we have

$$u(x,t) \leqslant \max\left\{\frac{A}{B}, \max_{x\in\overline{\Omega}} u_0(x)\right\} \text{ and } \limsup_{t\to+\infty} u(x,t) \leqslant \frac{A}{B}.$$

3. The existence and uniqueness of solution

To investigate the existence and boundedness of solutions of system (2)–(4), we introduce some notations.

Let $X = C\left(\overline{\Omega}, \mathbb{R}^2\right)$ be the Banach space of continuous functions from $\overline{\Omega}$ into $\mathbb{R}^2$ and $\mathcal{C} = C([-\tau,0],X)$ be the Banach space of continuous functions of $[-\tau,0]$ into X with standard uniform topology. For simplicity, we identify an element $\phi \in \mathcal{C}$ as a function from $[-\tau,0] \times \overline{\Omega}$ into $\mathbb{R}^2$ defined by $\phi(s,x) = \phi(s)(x)$. For any continuous function $\omega(\cdot)\colon [-\tau,b) \to X$ for $b > 0$, we set $\omega_t \in \mathcal{C}$ by $\omega_t(s) = \omega(t+s)$ for $s \in [-\tau,0]$.

As in [6], we assume that the general investment function $I(Y,K)$ satisfies the following hypothesis:
(H_1) There exist two constants $A > 0$ and $\bar{q} \geqslant 0$ such that $|I(Y,K) + \bar{q}K| \leqslant A$ for all $Y,K \in \mathbb{R}$.

Also, we assume that the variable depreciation $\delta(K)$ satisfies the following hypothesis:
(H_2) There exists $\delta_1 > 0$ such that $\delta(K) \geqslant \delta_1$ for all $K \in \mathbb{R}$.

Based on these hypotheses, we have the following result.

Theorem 1. *If (H_1) and (H_2) hold, then for any given initial $\Phi \in \mathcal{C}$ there exists a unique solution of problem (2)–(4) defined on $[0,+\infty)$. Furthermore, this solution is uniformly bounded.*

Proof. For each $\varphi = (\varphi_1, \varphi_2)^T \in \mathcal{C}$ and $x \in \overline{\Omega}$, we define $Z = (Z_1, Z_2)\colon \mathcal{C} \to X$ by

$$Z_1(\varphi)(x) = \alpha[I(\varphi_1(0,x), \varphi_2(0,x)) - \gamma\varphi_1(0,x)],$$
$$Z_2(\varphi)(x) = I(\varphi_1(-\tau_1,x), \varphi_2(-\tau_2,x)) - \delta(\varphi_2(0,x))\varphi_2(0,x).$$

Then problem (2)–(4) can be rewritten as the following abstract functional differential equation

$$\begin{cases} u'(t) = Eu(t) + Z(u_t), \quad t > 0, \\ u(0) = \phi \in \mathcal{C}, \end{cases} \tag{5}$$

where $u = (Y, K)^T$ and $Eu = (d_1\Delta Y, d_2\Delta K)^T$. It is obvious that Z is locally Lipschitz in $\mathcal{C}$, and as in [8], we conclude that problem (5) has a unique local solution on $[0, T_{\max})$, where $T_{\max}$ is the maximal existence time for solution of system (5).

From the second equation of (2) and $(H_1) - (H_2)$, we get

$$\begin{cases} \dfrac{\partial K}{\partial t} - d_2\Delta K \leqslant A - (\delta_1 + \bar{q})K, \\ \dfrac{\partial K}{\partial \xi} = 0, \\ K(0,x) = \Phi_2(0,x), x \in \bar{\Omega}. \end{cases}$$

According to Lemma 1, we have

$$K(t,x) \leqslant \max\left\{\frac{A}{\delta_1 + \overline{q}}, \max_{x\in\overline{\Omega}} \Phi_2(0,x)\right\}, \ \forall (t,x) \in [0, T_{\max}) \times \overline{\Omega}.$$

This implies that K is bounded.

According to the first equation of the system (2), we have

$$\begin{cases} \dfrac{\partial Y}{\partial t} - d_1\Delta Y \leqslant \varrho - \alpha\gamma Y, \\ \dfrac{\partial Y}{\partial \xi} = 0, \\ Y(0,x) = \Phi_1(0,x), \quad x \in \overline{\Omega}. \end{cases}$$

Similarly to above, we obtain

$$Y(t,x) \leqslant \max\left\{\frac{\varrho}{\alpha\gamma}, \max_{x\in\overline{\Omega}} \phi_1(0,x)\right\}, \ \forall (t,x) \in [0, T_{\max}) \times \overline{\Omega},$$

where $\varrho = \alpha(A + \overline{q}\nu)$ with $\nu = \max\{\frac{A}{\delta_1+\overline{q}}, \max\limits_{x\in\overline{\Omega}} \phi_2(0,x)\}$, which implies that Y is bounded. It follows from the standard theory for semilinear parabolic systems [9] that $T_{\max} = +\infty$. This completes the proof. ■

4. The economic equilibrium and its stability

4.1. The economic equilibrium

In the order to investigate the existence of equilibria of (2), we consider the following hypotheses:

(H_3) $I(0,0) > 0$ for all $Y \geqslant 0$, $K \geqslant 0$;

(H_4) $\frac{\delta'(K)K+\delta(K)}{\gamma}\frac{\partial I}{\partial Y} - \frac{\gamma\delta'(K)K+\gamma\delta(K)}{\gamma} + \frac{\partial I}{\partial K} < 0$ for all $Y \geqslant 0$, $K \geqslant 0$.

Theorem 2. *If (H_1)–(H_4) hold, then system (2) has an unique economic equilibrium of the form $E^*\big(\frac{\delta(K^*)K^*}{\gamma}, K^*\big)$, where K^* is the unique solution of the equation $I(\frac{\delta(K)K}{\gamma}, K) - \delta(K)K = 0$.*

Proof. Economic equilibrium is the solution of the following system:

$$\begin{cases} \alpha[I(Y,K) - \gamma Y] = 0, \\ I(Y,K) - \delta(K)K = 0. \end{cases} \tag{6}$$

Then

$$Y = \frac{\delta(K)K}{\gamma}. \tag{7}$$

Substituting (7) in (6), we find

$$I\left(\frac{\delta(K)K}{\gamma},K\right)-\delta(K)\,K=0.$$

Let V be the function defined on the interval $[0,+\infty)$ by

$$V(K)=I\left(\frac{\delta(K)K}{\gamma},K\right)-\delta(K)\,K.$$

Using the assumptions (H_1)–(H_4), we have $V(0)=I(0,0)>0$, $\lim\limits_{K\to+\infty}V(K)=-\infty$ and

$$V'(K)=\frac{\delta'(K)K+\delta(K)}{\gamma}\frac{\partial I}{\partial Y}-\frac{\gamma\delta'(K)K+\gamma\delta(K)}{\gamma}+\frac{\partial I}{\partial K}<0.$$

Therefore, there is a unique economic equilibrium $E^*(Y^*,K^*)$ where K^* is the solution of the equation $V(K)=0$ and $Y^*=\frac{\delta(K^*)K^*}{\gamma}$. ∎

4.2. Stability analysis and Hopf bifurcation

In this section, we study the stability analysis of the economic equilibrium and the existence of Hopf bifurcation.

Let $\mathcal{Y}=Y-Y^*$ and $\mathcal{K}=K-K^*$. By substituting y and k into system (2) and linearizing, we get the following system

$$\begin{cases}\dfrac{\partial\mathcal{Y}(t,x)}{\partial t}=d_1\Delta\mathcal{Y}(t,x)+\alpha[a\mathcal{Y}(t,x)+b\mathcal{K}(t,x)-\gamma\mathcal{Y}(t,x)],\\ \dfrac{\partial\mathcal{K}(t,x)}{\partial t}=d_2\Delta\mathcal{K}(t,x)+a\mathcal{Y}(t-\tau_1,x)+b\mathcal{K}(t-\tau_2)-\bar{\delta}\mathcal{K}(t),\\ \dfrac{\partial\mathcal{Y}}{\partial\xi}=\dfrac{\partial\mathcal{K}}{\partial\xi}=0,\quad t>0,\quad x\in\partial\Omega,\end{cases}\tag{8}$$

where $a=\frac{\partial I}{\partial Y}(Y^*,K^*)$, $b=\frac{\partial I}{\partial K}(Y^*,K^*)$ and $\bar{\delta}=K^*\delta'(K^*)+\delta(K^*)$.

Let $\zeta=C([-\tau,0],\mathbb{S})$ be the Banach space of continuous functions of $[-\tau,0]$ into $\mathbb{S}$, where $\mathbb{S}$ is defined by

$$\mathbb{S}=\left\{\mathcal{Y},\mathcal{K}\in W^{2,2}(\Omega)\colon\frac{\partial\mathcal{Y}(t,x)}{\partial\xi}=\frac{\partial\mathcal{K}(t,x)}{\partial\xi}=0,\ x\in\partial\Omega\right\}.$$

Then (8) can be rewritten as follows

$$U'(t)=\mathcal{D}\Delta U+L(U_t),$$

where $U=(\mathcal{Y},\mathcal{K})^T$, $\mathcal{D}=\operatorname{diag}(d_1,d_2)$ and $L\colon\zeta\to\mathbb{S}$ defined by

$$L(\phi)=\mathcal{A}\phi(0)+\mathcal{B}\phi(-\tau),$$

with

$$\mathcal{A}=\begin{pmatrix}\alpha(a-\gamma)&\alpha b\\0&-\bar{\delta}\end{pmatrix}\ \text{ and }\ \mathcal{B}=\begin{pmatrix}0&0\\a&b\end{pmatrix}.$$

Then the characteristic of system (10) is as follows

$$\lambda y-\mathcal{D}\Delta y-L\big(e^{\lambda}y\big)=0,\quad y\in\operatorname{dom}(\Delta)\backslash\{0\}.\tag{9}$$

Let $-k^2$ ($k\in\mathbb{N}=\{0,1,2,\dots\}$) be the eigenvalue of the operator Δ under the Neumann boundary conditions on $\mathbb{S}$ and the corresponding eigenvectors take the form:

$$\beta_k^1=(\sigma_k,0)^T,\quad\beta_k^2=(0,\sigma_k)^T,\quad\sigma_k=\cos(kx),\quad k=0,1,2,\dots$$

and $\{\beta_k^1,\beta_k^2\}_{k=0}^{+\infty}$ construct a basis of the phase space $\mathbb{S}$. Hence, we can expand in the form of Fourier on the phase space $\mathbb{S}$, which is as follows

$$y=\sum_{k=0}^{\infty}Y_k^T\begin{pmatrix}\beta_k^1\\\beta_k^2\end{pmatrix},\quad Y_k=\begin{pmatrix}\langle y,\beta_k^1\rangle\\\langle y,\beta_k^2\rangle\end{pmatrix}.\tag{10}$$

Thus,

$$L\left(\phi^T\begin{pmatrix}\beta_k^1\\\beta_k^2\end{pmatrix}\right)=L(\phi)^T\begin{pmatrix}\beta_k^1\\\beta_k^2\end{pmatrix},\quad k\in\mathbb{N}.\tag{11}$$

Substituting (11) and (10) into (9), we can have

$$\sum_{k=0}^{\infty} Y_k^T \left[\left(\lambda I_2 + \mathcal{D}k^2\right) - \begin{pmatrix} \alpha(a-\gamma) & \alpha\, b \\ a\, e^{-\lambda\tau_1} & \beta\, e^{-\lambda\tau_2} - \bar{\delta} \end{pmatrix} \right] \begin{pmatrix} \beta_k^1 \\ \beta_k^2 \end{pmatrix} = 0.$$

The characteristic equation of (11) is as follows:

$$\begin{vmatrix} \lambda + d_1k^2 - \alpha(a-\gamma) & -\alpha\, b \\ -a\, e^{-\lambda\tau_1} & \lambda + d_2k^2 - b\, e^{-\lambda\tau_2} + \bar{\delta} \end{vmatrix} = 0,$$

which leads to

$$\lambda^2 + p_{1,k}\lambda + p_{0,k} + q_1 e^{-\lambda\tau_1} + (r_1\lambda + r_{0,k})e^{-\lambda\tau_2} = 0, \tag{12}$$

where

$$p_{0,k} = [d_1k^2 - \alpha(a-\gamma)](d_2k^2 + \bar{\delta}), \qquad p_{1,k} = (d_1+d_2)k^2 - \alpha(a-\gamma) + \bar{\delta},$$
$$q_1 = -\alpha\, a\, b, \qquad r_1 = -b,$$
$$r_{0,k} = -b[d_1k^2 - \alpha(a-\gamma)].$$

4.3. The case $\tau_1 = \tau_2 = 0$

The Eq. (12) reduces to

$$\lambda^2 + c_{1,k}\lambda + c_{0,k} = 0, \tag{13}$$

with

$$c_{0,k} = p_{0,k} + q_1 + r_{0,k}, \quad c_{1,k} = p_{1,k} + r_1.$$

If $a < \gamma$, then the coefficients of the equation (13) satisfy:

$$c_{1,k} > 0 \quad \text{and} \quad c_{0,k} > 0.$$

Therefore, we have the following lemma.

Lemma 2. *If $a < \gamma$, then the economic equilibrium E^* is locally asymptotically stable in the case $\tau_1 = \tau_2 = 0$.*

4.4. The case $\tau_1 \neq 0$ and $\tau_2 = 0$

The Eq. (12) becomes

$$\lambda^2 + d_{1,k}\lambda + d_{0,k} + q_1 e^{-\lambda\tau_1} = 0, \tag{14}$$

with

$$d_{0,k} = p_{0,k} + r_{0,k}, \quad d_{1,k} = p_{1,k} + r_1.$$

Let $\lambda = \mathrm{i}\,\omega$ ($\omega > 0$) be a purely imaginary root of the equation (14). Then

$$-\omega^2 + \mathrm{i}\, d_{1,k}\omega + d_{0,k} + q_1 e^{-\mathrm{i}\,\omega\tau_1} = 0.$$

Hence,

$$\begin{cases} \omega^2 - d_{0,k} = q_1\cos(\omega\tau_1), \\ \omega\, d_{1,k} = q_1\sin(\omega\tau_1), \end{cases}$$

which implies that

$$\omega^4 + (d_{1,k}^2 - 2d_{0,k})\omega^2 + d_{0,k}^2 - q_1^2 = 0. \tag{15}$$

Let $z = \omega^2$. Thus, the equation (15) becomes

$$f(z) = z^2 + (d_{1,k}^2 - 2d_{0,k})z + d_{0,k}^2 - q_1^2 = 0. \tag{16}$$

By calculations, we obtain

$$d_{1,k}^2 - 2d_{0,k} = [d_1k^2 - \alpha(a-\gamma)]^2 + [d_2k^2 - (b-\bar{\delta})]^2 > 0,$$
$$d_{0,k}^2 - q_1^2 = [d_1k^2 - \alpha(a-\gamma)]^2[d_2k^2 - (b-\bar{\delta})]^2 - \alpha^2a^2b^2.$$

When $k=0$, it is easy to show that

$$d_{0,0}^2 - q_1^2 = \alpha^2(a-\gamma)^2(b-\bar{\delta})^2 - \alpha^2 a^2 b^2.$$

Clearly, if $|a-\gamma|(\bar{\delta}-b) \geqslant -ab$, then Eq. (16) has no positive root.

However, if $(\mathcal{A}_1)$: $|a-\gamma|(\bar{\delta}-b) < -ab$, then Eq. (16) has a unique positive root z_0 and thus Eq. (15) has a positive root $\omega_0 = \sqrt{z_0}$. In this case, we have

$$\tau_{1,j} = \frac{1}{\omega_0} \arccos\left(\frac{\omega_0^2 - d_{0,0}}{q_1}\right) + \frac{2j\pi}{\omega_0}, \quad j = 0, 1, 2, \dots,$$

at which Eq. (14) with $k=0$ has a pair of purely imaginary roots of the form $\pm \mathrm{i}\,\omega_0$ and all roots of Eq. (14), except $\pm \mathrm{i}\,\omega_0$, have no zero real parts. Then, by the general theory on characteristic equations of delay differential equations from [10] (Theorem 4.1), we see that if $a < \gamma$ and (A_1) hold, E^* remains stable for $\tau_1 < \tau_{1,0}$.

Let $\lambda(\tau_1) = v(\tau_1) + \mathrm{i}\,\omega(\tau_1)$ be a root of Eq. (14) satisfying $v(\tau_{1,0}) = 0$, $\omega(\tau_{1,0}) = \omega_0$. We now verify that

$$\left.\frac{d(\operatorname{Re}\lambda)}{d\tau}\right|_{\tau=\tau_{1,0}} > 0.$$

This will prove that there exists at least one eigenvalue with positive real part for $\tau > 1, 0$. In addition, the conditions for the existence of a Hopf bifurcation [10] are then satisfied yielding a periodic solution. To this end, differentiating Eq. (14) with respect τ_1, we derive that

$$\left(\frac{d\lambda}{d\tau}\right)^{-1} = -\frac{2\lambda + d_{1,k}}{\lambda\left(\lambda^2 + d_{1,k}\lambda + d_{0,k}\right)} - \frac{\tau_1}{\lambda}.$$

By direct calculations one obtain that

$$\begin{aligned}
\operatorname{sign}\left\{\frac{d(\operatorname{Re}\lambda)}{d\tau}\right\}_{\lambda=\mathrm{i}\,\omega_0} &= \operatorname{sign}\left\{\operatorname{Re}\left(\frac{d\lambda}{d\tau}\right)^{-1}\right\}_{\lambda=\mathrm{i}\,\omega_0} \\
&= \operatorname{sign}\left\{\frac{2\omega_0^2 + 2d_{1,k}^2 - 2d_{0,k}}{(d_{0,k}-\omega_0^2)^2 + \omega_0^2 d_{1,k}^2}\right\} \\
&= \operatorname{sign}\left\{\frac{f'(z_0)}{(d_{0,k}-\omega_0^2)^2 + \omega_0^2 d_{1,k}^2}\right\},
\end{aligned}$$

where $z_0 = \omega_0^2$. Hence, the transversal condition holds and a Hopf bifurcation occurs at $\omega = \omega_0$, $\tau_1 = \tau_{1,0}$.

In addition, if $i\omega_1$ is a solution of (14) with $k \geqslant 1$, we get

$$\omega_1^4 + (d_{1,k}^2 - 2d_{0,k})\omega_1^2 + d_{0,k}^2 - q_1^2 = 0.$$

On the other hand, we have

$$d_{0,k}^2 - q_1^2 > [d_1 - \alpha(a-\gamma)]^2[d_2 - (b-\bar{\delta})]^2 - \alpha^2 a^2 b^2, \text{ for all } k \geqslant 1.$$

If $(\mathcal{A}_2)$: $|d_1 - \alpha(a-\gamma)|\left(d_2 + \bar{\delta} - b\right) > -\alpha ab$, then Eq. (14) with $k \geqslant 1$ has no purely imaginary roots. Therefore, we have the following result.

Theorem 3. *For $\tau_2 = 0$, we have the following conclusion.*
If $a < \gamma$ and $(\mathcal{A}_1) - (\mathcal{A}_2)$ hold, then the economic equilibrium E^* is locally asymptotically stable for $\tau_1 < \tau_{1,0}$ and unstable for $\tau_1 > \tau_{1,0}$. In addition, the system (1) undergoes Hopf bifurcation at E^* when $\tau_1 = \tau_{1,0}$, where $\tau_{1,j} = \frac{1}{\omega_0}\arccos\left(\frac{\omega_0^2 - d_{0,0}}{q_1}\right) + \frac{2j\pi}{\omega_0}$, $j \in \mathbb{N}$.

4.5. The case $\tau_1 \neq 0$, $\tau_2 \neq 0$

In this case, we consider Eq. (12) with $\tau_2 > 0$ and τ_1 in the stable regions. Regards τ_2 as a parameter of bifurcation. From Ruan and Wei [11], we have the following result.

Lemma 3. *If all roots of equation (14) have negative real parts for $\tau_1 > 0$, then there exists a $\tau_2^*(\tau_1) > 0$ such that when $0 \leqslant \tau_2 < \tau_2^*(\tau_1)$ all roots of equation (12) have negative real parts.*

Proof. The left hand side of Eq. (12) is analytic in λ and τ_2. From [11], we deduce when τ_2 varies, the sum of the multiplicities of zeros of the left hand side of equation Eq. (12) in the open right half-plane can change only if a zero on or cross the imaginary axis. ∎

Theorem 4. *For τ_1 in the stable regions and $\tau_2 > 0$, we have*

(i) If $f(z) = 0$ has no positive roots, there exists a $\tau_2^*(\tau_1)$ such that the economic equilibrium E^* is locally asymptotically stable when $\tau_2 \in [0, \tau_2^*(\tau_1))$.

(ii) If $a < \gamma$ and $(\mathcal{A}_1) - (\mathcal{A}_2)$ hold, then for any $\tau_1 \in [0, \tau_{1,0})$, there exists a $\tau_2^*(\tau_1)$ such that the economic equilibrium E^* is locally asymptotically stable when $\tau_2 \in [0, \tau_2^*(\tau_1))$.

Proof. The proof of (i) is immediate from Lemma 3 and Theorem 3 (i).

Now, we prove (ii). When $a < \gamma$ and $(\mathcal{A}_1) - (\mathcal{A}_2)$ are satisfied, it follows from Theorem 3 that the economic equilibrium is locally asymptotically stable for $\tau_1 \in [0, \tau_{1,0})$. Then all roots of Eq. (14) have negative real parts. According to Lemma 3, there exists a $\tau_2^*(\tau_1) > 0$, such that when $0 \leqslant \tau_2 < \tau_2^*(\tau_1)$ all roots of equation (12) have negative real parts. Hence, the economic equilibrium E^* is locally asymptotically stable when $\tau_2 \in [0, \tau_2^*(\tau_1))$. ∎

It is clear that Hopf bifurcation occurs at $\tau_2^*(\tau_1)$ if the conditions of Lemma 3 or Theorem 4 are satisfied.

4.6. Study of special case

In this subsection, we consider the following business cycle model:

$$\begin{cases} \dfrac{dY}{dt} = d_1 \Delta Y(t,x) + \alpha[I(Y(t), K(t)) - \gamma Y(t)], \\ \dfrac{dK}{dt} = d_2 \Delta K(t,x) + I(Y(t-\tau), K(t-\tau)) - \delta(K)K(t). \end{cases} \tag{17}$$

This system is a particular case of system (2) with $\tau_1 = \tau_2 = \tau$. From Theorems 1 and 2, we have the following results.

Corollary 1.

(i) If (H_1) and (H_2) hold, then for any initial condition $(\Phi_1, \Phi_2) \in \mathcal{C}$, there exists a unique solution of system (17) defined on $[0, +\infty)$ and this solution is uniformly bounded.

(ii) If (H_1)–(H_4) hold, then system (17) has a unique economic equilibrium of the form $E^*\big(\frac{\delta(K^*)K^*}{\gamma}, K^*\big)$, where K^* is the unique solution of the equation $I\big(\frac{\delta(K)K}{\gamma}, K\big) - \delta(K)K = 0$.

Next, we discuss the stability analysis of system (17). In this case, Eq. (12) becomes

$$\lambda^2 + p_{1,k}\lambda + p_{0,k} + (q_1 + r_1\lambda + r_{0,k})e^{-\lambda\tau} = 0. \tag{18}$$

When $\tau = 0$, all roots of Eq. (18) have negative real parts if $a < \gamma$. Hence, E^* is locally asymptotically stable.

For $\tau > 0$, let $\mathrm{i}\omega$ $(\omega > 0)$ is a root of (18), then we have

$$\begin{cases} -\omega^2 + p_{0,k} = -(q_1 + r_{0,k})\cos(\omega\tau) - \omega r_1 \sin(\omega\tau), \\ \omega p_{1,k} = (q_1 + r_{0,k})\sin(\omega\tau) - \omega r_1 \cos(\omega\tau), \end{cases}$$

which leads to

$$\omega^4 + \big[p_{1,k}^2 - 2p_{0,k} - r_1^2\big]\omega^2 + \big[p_{0,k}^2 - (q_1 + r_{0,k})^2\big] = 0. \tag{19}$$

Letting $z = \omega^2$, Eq. (19) can be written as

$$g(z) = z^2 + \big[p_{1,k}^2 - 2p_{0,k} - r_1^2\big]z + \big[p_{0,k}^2 - (q_1 + r_{0,k})^2\big] = 0. \tag{20}$$

Clearly, if $a < \gamma$ and $g(z)$ has no positive roots, then the economic equilibrium E^* of (17) is locally asymptotically stable for all $\tau \geqslant 0$. If not, for certain $k_0 \in \mathbb{N}$, if Eq.(20) has positive roots, preserving generality, we suppose that Eq. (20) with $k = k_0$ has two positive roots z_n $(n = 1, 2)$. Hence, Eq. (19)

has two positive roots $\omega_n = \sqrt{z_n}$ and there exist two sequences of critical values of τ given by

$$\tau_j^n = \frac{1}{\omega_n} \arccos\left(\frac{\omega_n^2(r_1 p_{1,k} + q_1 + r_{0,k}) - p_{0,k}(q_1 + r_{0,k})}{(q_1 + r_{0,k})^2 - r_1\omega_n^2}\right) + \frac{2j\pi}{\omega_n},$$

where $j \in \mathbb{N}$.

Let $\lambda(\tau) = \sigma(\tau) + \mathrm{i}\,\omega(\tau)$ be the root of Eq. (18) satisfying $\sigma(\tau_j^n) = 0$ and $\omega(\tau_j^n) = \omega_n$. Differentiating Eq. (18) with respect to τ, we get

$$\left(\frac{d\lambda}{d\tau}\right)^{-1} = -\frac{2\lambda + p_{1,k}}{\lambda(\lambda^2 + p_{1,k}\lambda + p_{0,k})} + \frac{r_1}{\lambda(r_1\lambda + q_1 + r_{0,k})} - \frac{\tau}{\lambda}.$$

Let

$$\tau_0^* = \tau_0^{n_0} = \min_{n\in\{1,2\}} \{\tau_0^n\}, \quad \omega_0^* = \omega_{n_0}.$$

By a simple calculation, we have

$$\begin{aligned}\operatorname{sign}\left\{\frac{d(\operatorname{Re}\lambda)}{d\tau}\right\}_{\tau=\tau_0^*} &= \operatorname{sign}\left\{\frac{2(\omega_0^{*2} - p_{0,k}) + p_{1,k}^2 - r_1^2}{r_1^2\omega_0^{*2} + (q_1 + r_{0,k})^2}\right\} \\ &= \operatorname{sign}\left\{\frac{g'(z_0^*)}{r_1^2\omega_0^{*2} + (q_1 + r_{0,k})^2}\right\} \neq 0,\end{aligned}$$

where $z_0^* = \omega_0^{*2}$. We can deduce the following conclusions.

Theorem 5. *Assume that $a < \gamma$. Then we have*

(i) If $g(z) = 0$ has no positive roots, then the economic equilibrium E^* of system (17) is locally asymptotically stable for all $\tau > 0$.

(ii) If $g(z) = 0$ has positive roots, then system (17) undergoes a Hopf bifurcation at E^* when $\tau = \tau_0^*$. Further, the economic equilibrium E^* is locally asymptotically stable when $\tau \in [0, \tau_0^*)$ and unstable for $\tau > \tau_0^*$.

5. Numerical simulation

This section presents numerical simulation for our analytical results. Here, we take $I(Y,K) = \frac{e^Y}{1+e^Y} + \frac{qK}{\sqrt{1+\varepsilon K^2}}$, where $q < 0$, $\varepsilon \geqslant 0$. The depreciation rate function is chosen as $\delta(K) = \delta_1 + \frac{\delta_0 - \delta_1}{1+K}$, where $0 < \delta_1 < \delta_0$. Let $\alpha = 3$, $q = -0.5$, $\varepsilon = 0.01$, $\delta_1 = 0.2$, $\delta_1 = 0.8$, $\gamma = 0.3$, $d_1 = d_2 = 0.1$. In this case, our model has an economic steady state $E^*(0.7652, 0.9096)$. By a simple computation, we can obtain $a = 0.1436$, $b = -0.4939$. Now, we examine the dynamics of the system (2) for different values of τ_1 and τ_2.

First, we choose $\tau_1 \neq 0$ and $\tau_2 = 0$. Based on Theorem 3, we deduce that the economic steady state E^* is locally asymptotically stable for all $\tau \in [0, \tau_{1,0})$ and it is unstable for all $\tau_1 > \tau_{1,0}$ with $\tau_{1,0} \approx 5.6531$. Figures 1, 2, 3 and 4 illustrate this finding. When $\tau_1 = \tau_{1,0}$, the model (2) undergoes a Hopf bifurcation at the economic steady state E^* according to Theorem 3. Figures 5 and 6 show such result.

Based on the preceding, the stable region for τ_1 is $I = [0, 5.6531)$. If we pick out $\tau_1 = \tau_1^* = 0.5 \in I$, then we get $\tau_2^* = 2.0521$. Based on Theorem 4 (ii), $E^*(0.7652, 0.9096)$ is locally asymptotically stable when $\tau_2 \in [0, \tau_2^*)$. Figures 7, 8, 9 and 10 illustrate this results.

For $\tau_1 = \tau_2 = \tau$, the conditions of Theorem 5 (ii) are established. Then system (2) undergoes a Hopf bifurcation at $E^*(0.7652, 0.9096)$ when $\tau = \tau_0^* \approx 1.837$. Further, E^* is locally asymptotically stable for all $\tau \in [0, \tau_0^*)$ and it is unstable for all $\tau > \tau_0^*$. This outcome is depicted in Figures 11–16.

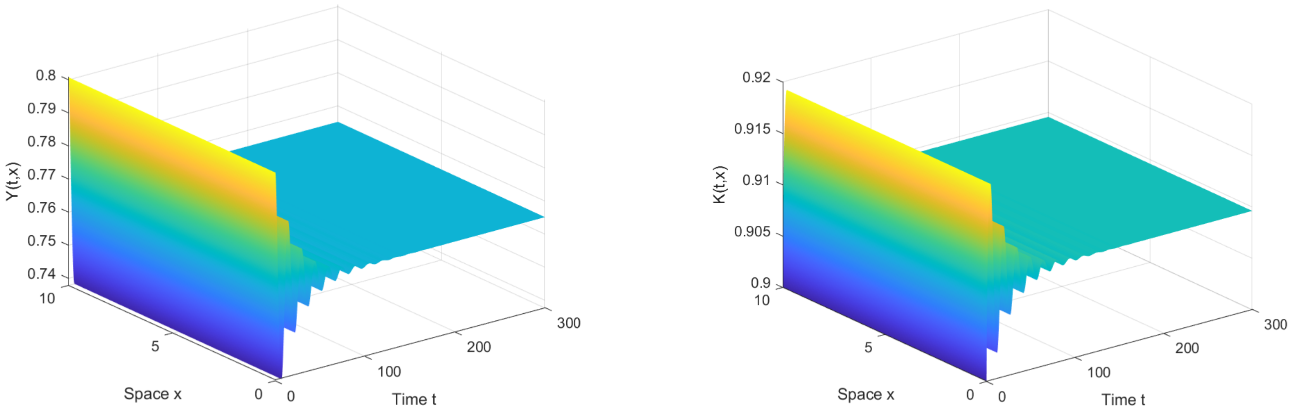

Fig. 1. Diffusive business cycle of system (2) when $\tau_1 = 1.5$ and $\tau_2 = 0$.

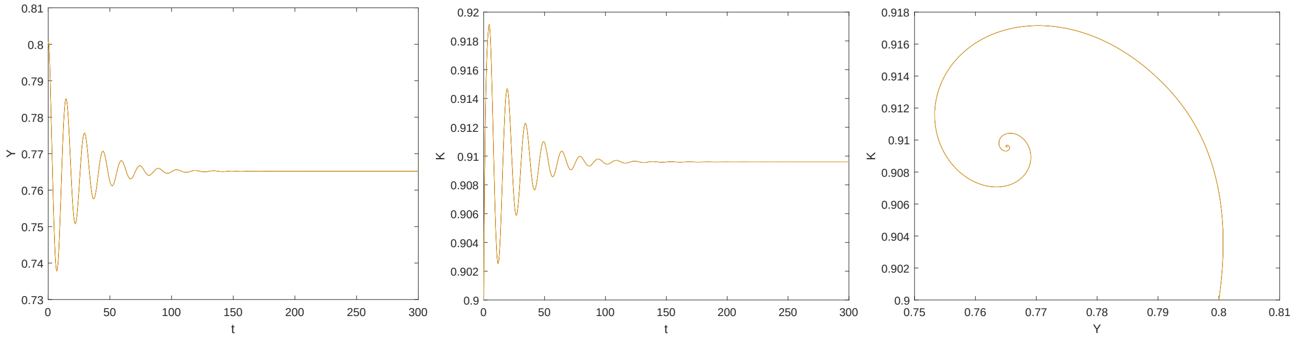

Fig. 2. The economic steady state E^* is stable when $\tau_1 = 1.5$, $\tau_2 = 0$ at $x = 5$.

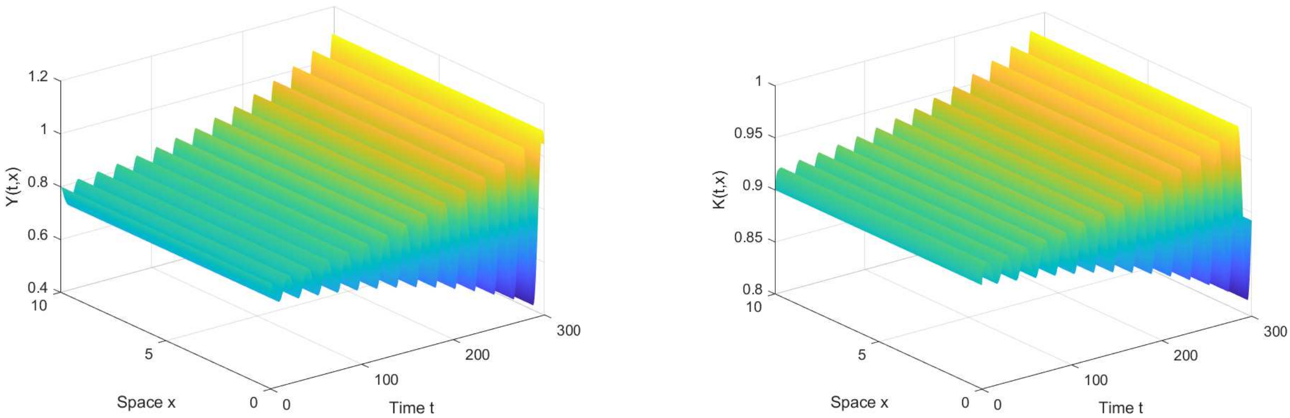

Fig. 3. Diffusive business cycle of system (2) when $\tau_1 = 6.5$ and $\tau_2 = 0$.

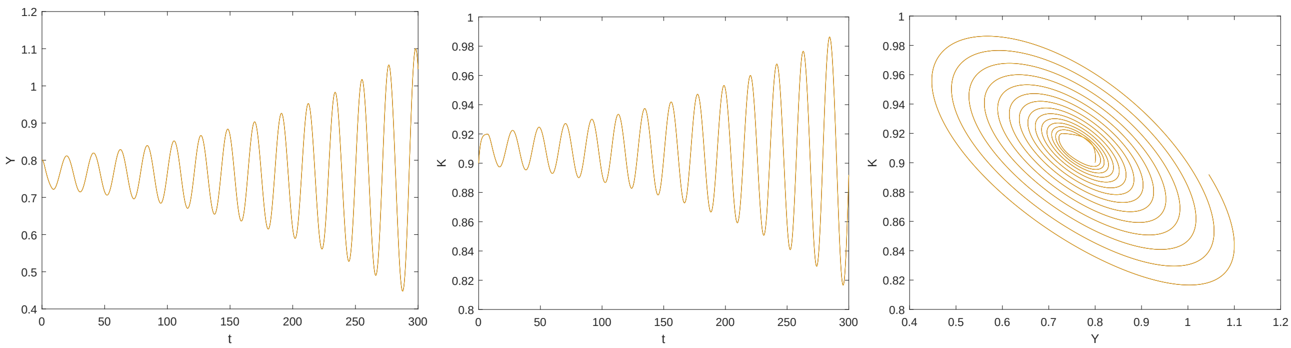

Fig. 4. The economic steady state E^* is unstable when $\tau_1 = 6.5$ and $\tau_2 = 0$ at $x = 5$.

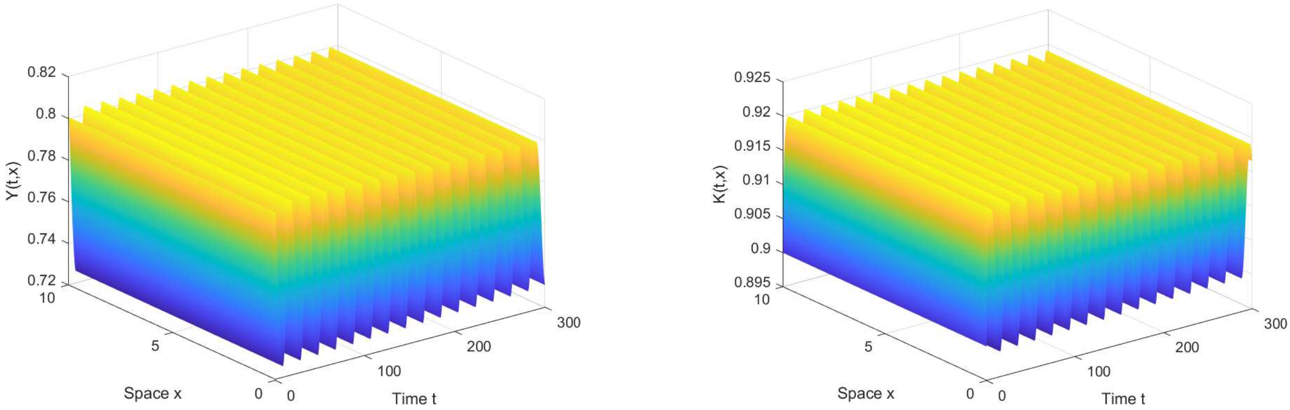

Fig. 5. Diffusive business cycle of system (2) $\tau_1 = \tau_{1,0}$ and $\tau_2 = 0$.

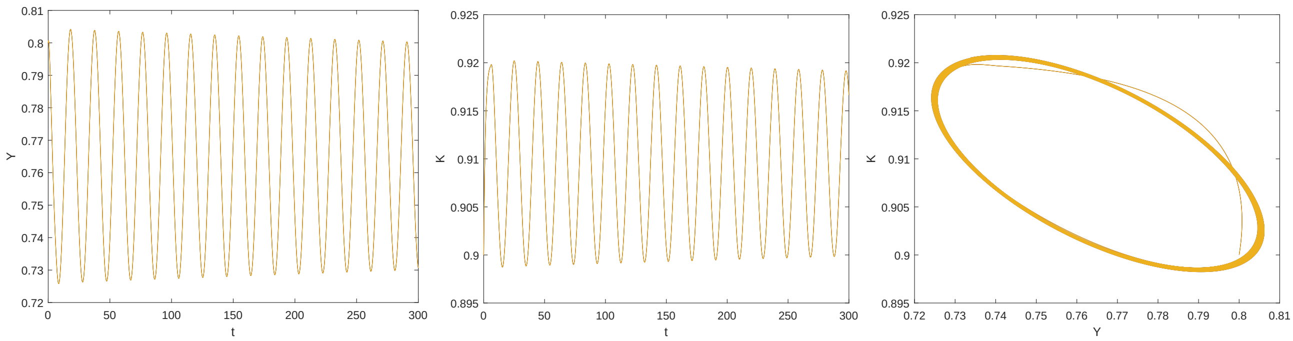

Fig. 6. System (2) undergoes Hopf bifurcation at the economic steady state E^* when $\tau_1 = \tau_{1,0}$ and $\tau_2 = 0$ at $x = 5$.

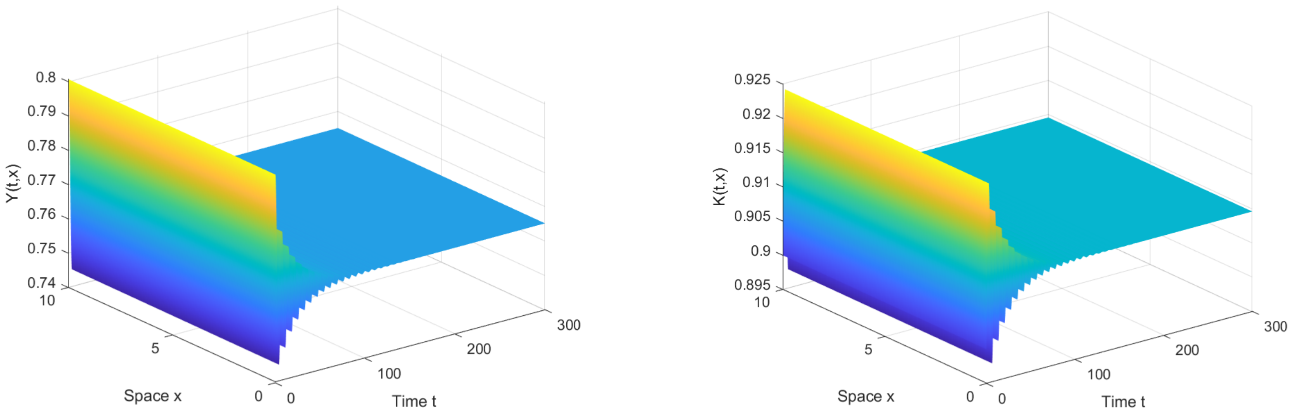

Fig. 7. Diffusive business cycle of system (2) when $\tau_1 = 0.5$ and $\tau_2 = 2$.

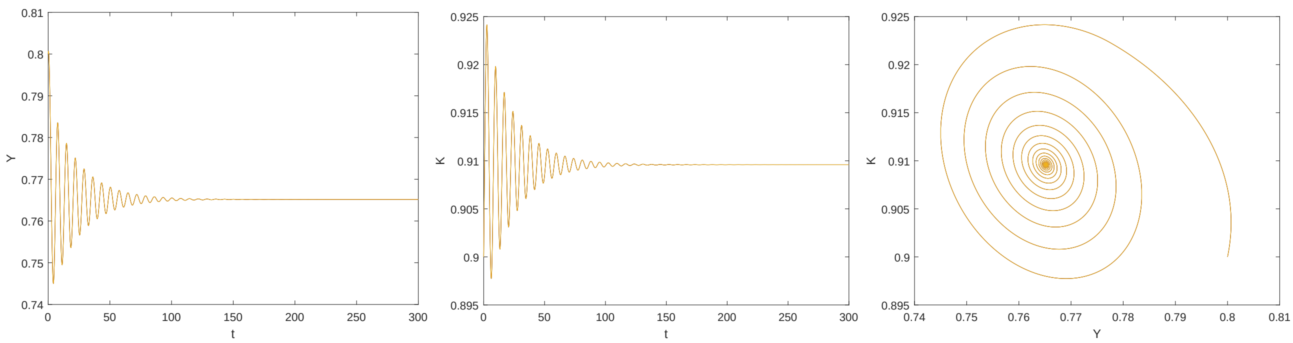

Fig. 8. The economic equilibrium E^* is stable when $\tau_1 = 0.5$ and $\tau_2 = 2$ at $x = 5$.

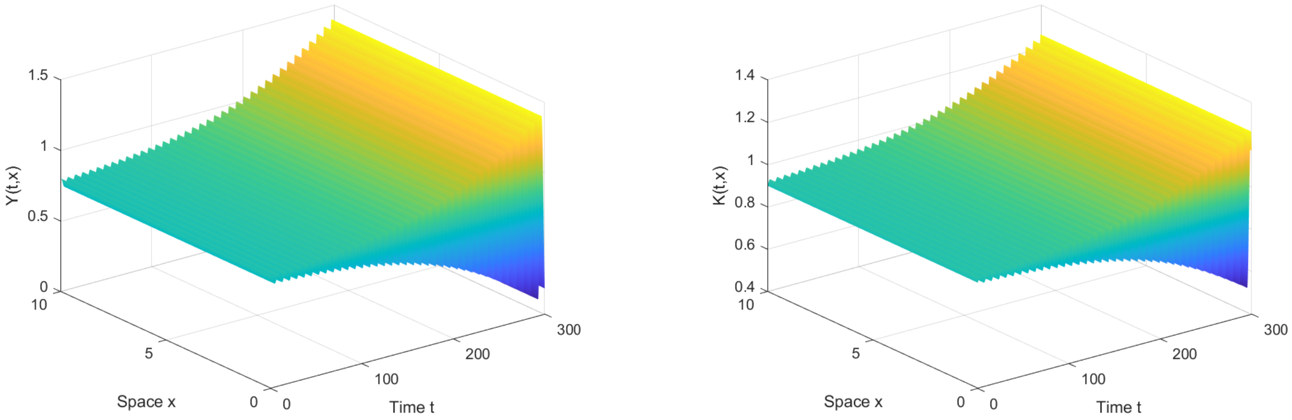

Fig. 9. Diffusive business cycle of system (2) when $\tau_1 = 0.5$ and $\tau_2 = 3.75$.

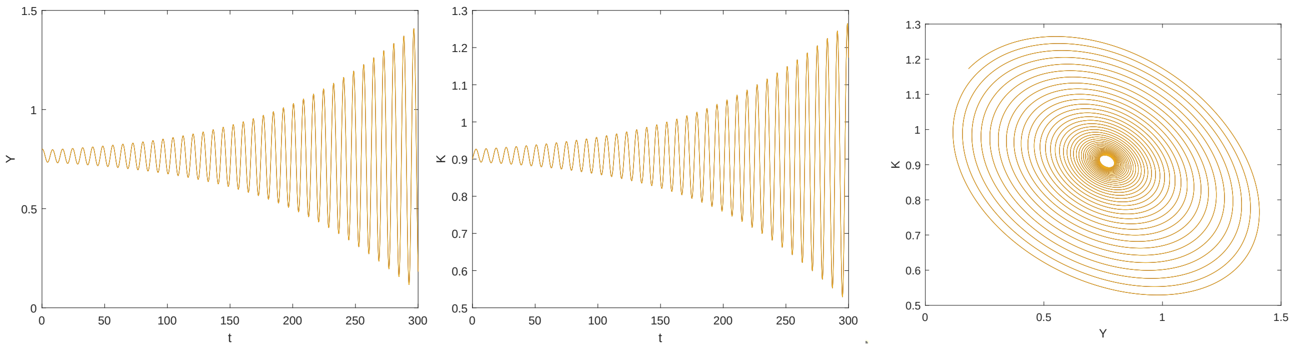

Fig. 10. The economic steady state E^* is unstable when $\tau_1 = 0.5$ and $\tau_2 = 3.75$ at $x = 5$.

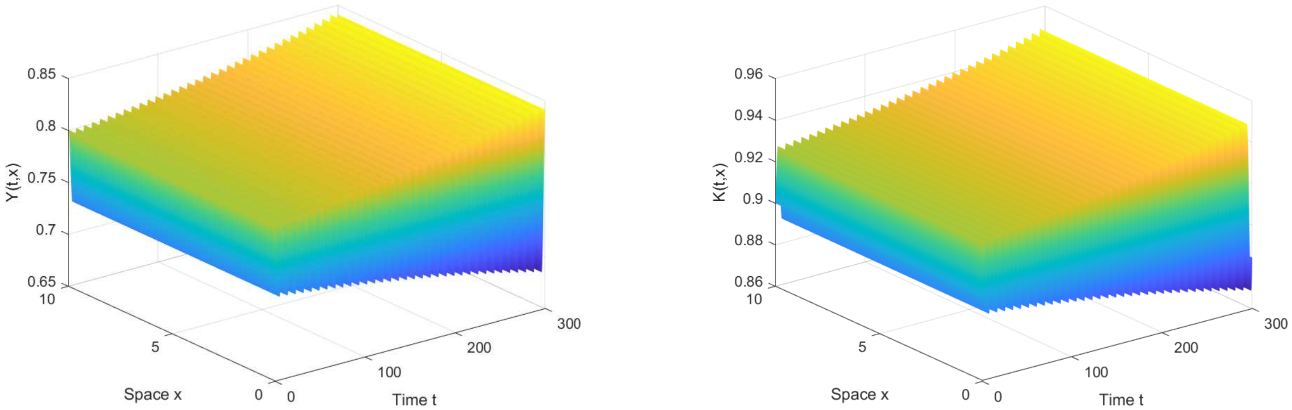

Fig. 11. Diffusive business cycle of system (2) when $\tau_1 = \tau_2 = 2$.

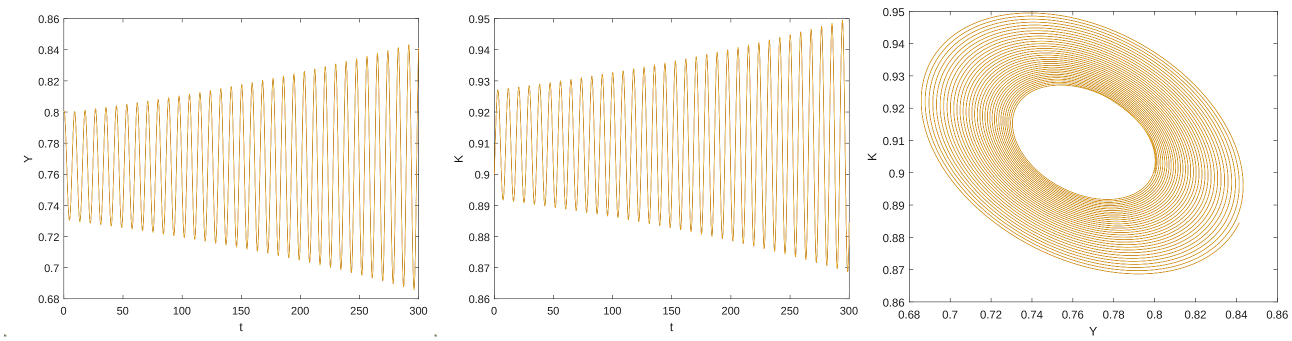

Fig. 12. The economic steady state E^* is unstable when $\tau_1 = \tau_2 = 2$ at $x = 5$.

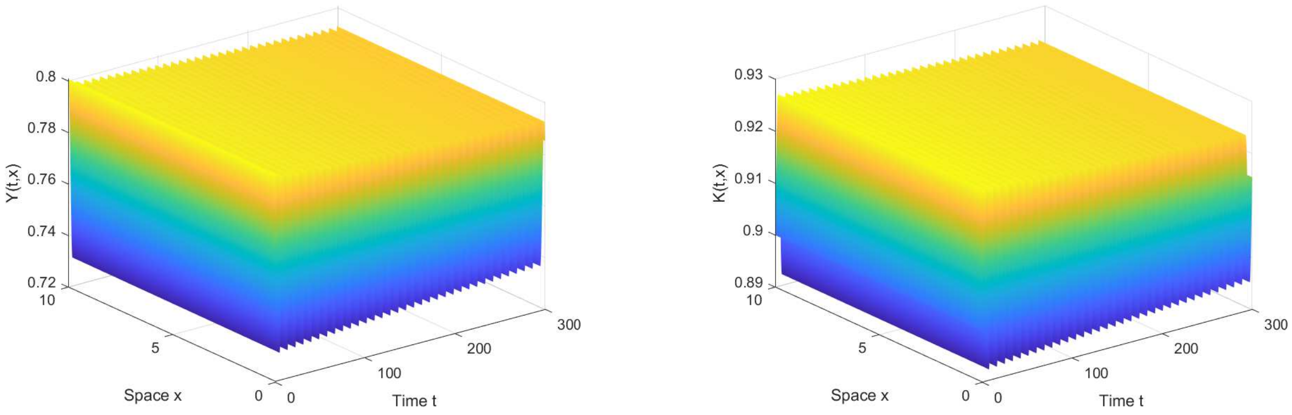

Fig. 13. Diffusive business cycle of system (2) when $\tau_1 = \tau_2 = \tau = \tau_0^*$.

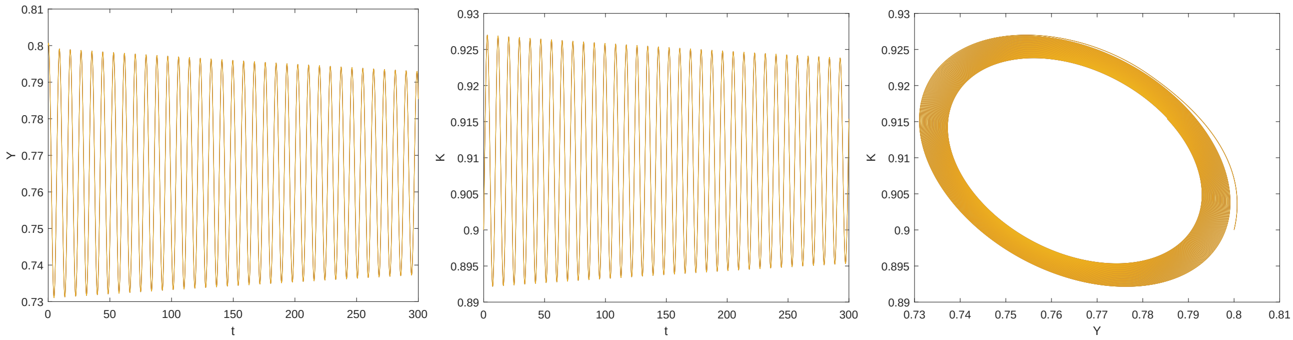

Fig. 14. System (2) undergoes Hopf bifurcation at the economic steady state E^* when $\tau_1 = \tau_2 = \tau_0^*$ at $x = 5$.

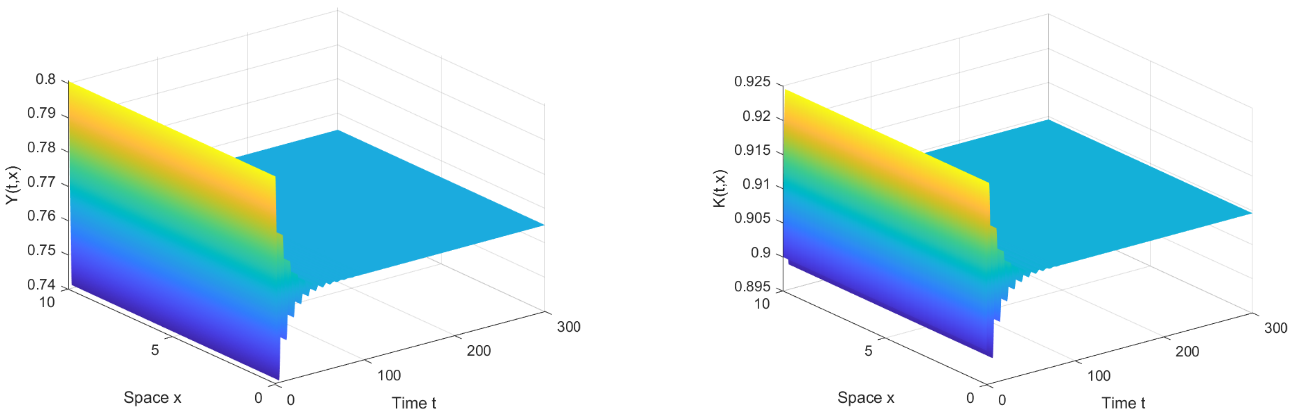

Fig. 15. Diffusive business cycle of system (2) when $\tau_1 = \tau_2 = 1.5$.

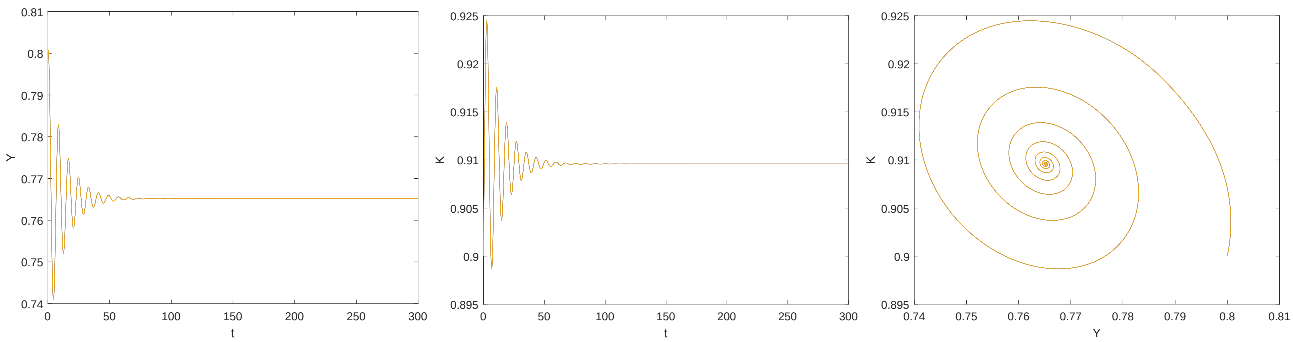

Fig. 16. The economic steady state E^* is stable when $\tau_1 = \tau_2 = 1.5$ at $x = 5$.

6. Conclusion

In this study, we have developed a mathematical model that describes the dynamics of delayed economic cycles with variable depreciation rate. This developed model was built by systems of PDEs to study the impact of diffusion effect. We firstly have demonstrated the existence and uniqueness of the solution and the economic equilibrium of our model. In addition, we have shown the local asymptotic stability under some conditions and prove that the introduction of delay can lead to a Hopf bifurcation at economic equilibrium. Finally, our numerical simulations confirm the theoretical results by illustrating the effect of delays on the spatiotemporal dynamics of our model and the occurrence of the Hopf bifurcation.

On the other side, memory is an important characteristic in economical systems that refers to the collective and historical knowledge, experiences and information that society has accumulated over time. Therefore, it very interesting to investigate the effect of memory on the spatiotemporal dynamics by means of Hattaf fractional and fractal fractional operators introduced in [12–14]. This well be the objective of our future research.

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Динаміка моделі дифузійного бізнес-циклу з двома затримками та змінною нормою амортизації

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Основною метою цієї статті є аналіз динаміки моделі відкладеного бізнес-циклу, яка описується рівняннями в частинних похідних (PDE), щоб врахувати швидкість амортизації основного капіталу та ефект дифузії. По-перше, ретельно вивчається існування розв'язків і економічна рівновага. По-друге, встановлено локальну стійкість та існування біфуркації Хопфа. Накінець, для ілюстрації аналітичних результатів представлено декілька чисельних симуляцій.

Ключові слова: *діловий цикл; норма амортизації; дифузійний ефект; біфуркація Хопфа; стійкість.*