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**Ihor Kuzio[1], Yurii Sholoviy[2], Nadiia Maherus[3]**

[1] Department of Robotics and Integrated Mechanical Engineering Technologies, Lviv Polytechnic National University, 12, S. Bandery str., Lviv, Ukraine, E-mail: ihor.v.kuzo@lpnu.ua, ORCID 0000-0001-9271-6505
[2] Department of Robotics and Integrated Mechanical Engineering Technologies, Lviv Polytechnic National University, 12, S. Bandery str., Lviv, Ukraine, E-mail: yurii.p.sholovii@lpnu.ua, ORCID 0000-0003-0154-7983
[3] Department of Robotics and Integrated Mechanical Engineering Technologies, Lviv Polytechnic National University, 12, S. Bandery str., Lviv, Ukraine, E-mail: nadiia.i.maherus@lpnu.ua, ORCID 0000-0003-1946-8503

# INVESTIGATION OF OSCILLATIONS IN A SYSTEM WITH NONLINEAR ELASTIC CHARACTERISTICS

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**Abstract.** Goal of the work is to apply periodic Ateb-functions to investigate dynamic processes of strongly nonlinear systems with a finite number of degrees of freedom. *Significance.* Practically all problems in mechanics and engineering related to system oscillations, when strictly formulated, are nonlinear, as they are mathematically described by nonlinear differential equations. This often poses significant challenges in studying their behaviour, as finding a solution in exact form can be difficult. One way to overcome this complexity is utilizing special functions, such as Ateb-functions. Therefore, the application of periodic Ateb-functions for investigating the dynamic processes of highly nonlinear systems with a finite number of degrees of freedom is a relevant task, as it allows for increased accuracy and efficiency of estimations. *Method.* The methodology is based on finding partial solutions of the "normal" oscillations form, which do not correspond to linearized systems. The normal modes of oscillation of highly nonlinear conservative systems, whose potential energy is a homogeneous function of degree $\nu+1$, are described using periodic Ateb-functions. In cases where linearization of the original system is not possible, such an approach to studying the oscillatory processes of highly nonlinear systems with multiple degrees of freedom is the most feasible. *Results.* The presented methodology for investigating normal modes of oscillations can be generalized to highly nonlinear systems with small perturbations of autonomous and non-autonomous types. Scientific novelty. Mathematical relations have been established to determine the normal modes of oscillations in highly nonlinear mechanical systems. *Practical significance.* The application of periodic Ateb-functions for investigating dynamic processes in highly nonlinear systems will enhance the accuracy and efficiency of estimations.

**Keywords:** oscillatory systems, dynamic processes, nonlinear systems, Ateb-functions, normal forms of oscillations.

## Introduction

Nowadays, the analysis of the dynamics of mechanical systems with a high degree of nonlinearity is a widely spread phenomenon. Typically, dynamic processes within systems with concentrated masses and distributed parameters are described by nonlinear differential equations. In the case of small oscillations of the system around the equilibrium state, the mathematical apparatus known in scientific arias today is sufficient for practical tasks. However, concerning systems with large oscillations, analytical methods of their study remain as a current issue.

## Review of Modern Information Sources on the Subject of the Paper

Analytical methods for investigating oscillatory processes in linear systems with both single and multiple degrees of freedom are extensively covered in the literature [1]. However, when it comes to nonlinear systems, difficulties often arise during studying these systems. A widely practiced approach is to linearize mechanical nonlinear oscillatory systems to investigate their behaviour [2, 3]. However, in general nearly all mechanic and engineering problems related to system oscillations are nonlinear, meaning they are mathematically described by nonlinear differential equations. Examples of such mechanical systems could include:

- rotary systems with nonlinear oscillations;
- vehicle equipment influenced to various mechanical loads;
- continuous transport machinery.

Analysis of nonlinear systems performing large oscillations has become widely popular in the last decades. These studies primarily rely on numerical methods to predict a system behaviour. Often, investigations into both self-oscillations and forced oscillations in nonlinear systems boil down to finding or determining conditions for an existence of stable periodic solutions of nonlinear differential equations that describe a motion of systems under the study [4]. In the initial stages of researching oscillatory systems, nonlinear processes were considered only in specific cases. One of the first mathematical methods, which was used to study nonlinear oscillatory systems, was a perturbation theory. However, this theory allowed to investigate the oscillatory of mechanical systems containing a so-called “small parameter”, and when its value is zero, such systems are studied as linear. However, alongside this, there are many systems where using a linear approach is inadmissible [5]. When studying oscillations in nonlinear systems, the “small parameter method” takes an important role, which was initiated in the works of Poincaré. The main idea of this method is as follows. There is a nonlinear system, whose motion is described by the differential equation:

$$\ddot{x} + \omega^2 x = \varepsilon f(x, \dot{x}) \quad (1)$$

with $\omega$ is a constant, $\varepsilon$ is a small parameter.

Taking into account that the period of oscillations of a nonlinear system depends on the small parameter, by a certain substitution of the independent variable $t$, the equation (1) is transformed into a form which the period of its solution becomes constant.

In the final analysis, the problem is reduced to solving linear differential equations with certain unknown parameters; however, the procedure for finding these parameters is not always simple and straightforward. The primary difficulty of mechanical systems with nonlinear oscillations lies in the possibility of finding a solution in an exact form. To address this, various approximate numerical methods have been developed, but there are also options for utilizing special functions: Lyapunov established new functions $C$ and $S$ [6]; Rosenberg constructed Ateb-functions [7]; Senik expressed these solutions as three-argument functions *ca* and *sa* [8, 9], which can be transformed into Jacobian elliptic functions when the nonlinearity is cubic. In the study [10], the ca function and the Jacobian elliptic function *cn* are used to investigate oscillatory systems with concentrated masses and distributed parameters.

## Research results

The subject of this article is to apply periodic Ateb-functions to the study of dynamic processes in strongly nonlinear systems with a finite number of degrees of freedom. This method is based on finding partial solutions in the form of “normal” oscillations, which do not correspond to linearized systems, but can still be found in a closed form for certain classes of multiple degree of freedom systems. When linearization of the original system is not possible, the most appropriate way is using an approach to investigating oscillatory processes in strongly nonlinear systems with multiple degrees of freedom.

Let’s consider a conservative system with a finite number of degrees of freedom. It is supposed its potential energy be a homogeneous function of the generalized coordinates $y_1,\ y_2,\ldots,y_n$ with the homogeneity exponent $v+1$, namely:

$$P(y_1,y_2,...,y_n)=\sum c_{r_1,r_2,...,r_n} y_1^{v_1} y_2^{v_2}...y_n^{v_n} \tag{2}$$

and the kinetic energy has a well-defined quadratic form:

$$T(\dot y_1,\dot y_2,...,\dot y_n)=\frac{1}{2}\sum_{i=1}^{n} m_i \dot y_i^{\,2} \tag{3}$$

with $v+1=\sum_{i=1}^{n} v_i$ , $c_{r_1,r_2,...,r_n}$ , $m_i$ are constants.

Then the differential equations of the system motion take the form:

$$\ddot y_i + m_i^{-1} v_i \sum c_{r_1,r_2,...,r_n} y_1^{v_1} y_2^{v_2}...y_{i-1}^{v_{i-1}} y_i^{v_i} y_{i+1}^{v_{i+1}}...y_n^{v_n} = 0 . \tag{4}$$

It is assumed that the system undergoes nonlinear oscillations that coincide in form with the first generalized coordinate (the “mode”), namely:

$$y_i = b_i y_1,\ i=2,3,...,n \tag{5}$$

with $b_i$ are some constants, the conditions for determining them will be specified below. Substituting this into the differential equations of motion, it is obtained:

$$\ddot y + m_1^{-1} y^v v_1 \sum c_{r_1,r_2,...,r_n} 1(b_2)^{v_2}...(b_n)^{v_n} = 0,$$
$$b_2\ddot y + m_2^{-1} y^v v_2 \sum c_{r_1,r_2,...,r_n} 1(b_2)^{v_2-1}(b_3)^{v_3}...(b_n)^{v_n} = 0,$$
$$b_3\ddot y + m_3^{-1} y^v v_3 \sum c_{r_1,r_2,...,r_n} 1(b_2)^{v_2}(b_3)^{v_3-1}(b_4)^{v_4}...(b_n)^{v_n} = 0,$$
$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots..$$
$$b_n\ddot y + m_n^{-1} y^v v_n \sum c_{r_1,r_2,...,r_n} 1(b_2)^{v_2}(b_3)^{v_3}(b_4)^{v_4}...(b_{n-1})^{v_{n-1}}...(b_n)^{v_n-1} = 0. \tag{6}$$

In the equations (6) and ones below, the subscript 1, which indicates the first “mode” of system oscillations, is omitted. The obtained differential equations differ only in the coefficients accompanying the derivative in the nonlinear term, so they can be written as:

$$\ddot y + b_i^0 y^v = 0 \tag{7}$$

with

$$b_1^0 = m_1^{-1} v_1 \sum c_{r_1,r_2,...,r_n} 1(b_2)^{v_2}...(b_n)^{v_n} ,$$
$$b_2^0 = m_2^{-1}\left(b_2\right)^{-1} v_2 \sum c_{r_1,r_2,...,r_n} 1(b_2)^{v_2-1}(b_3)^{v_3}...(b_n)^{v_n},$$
$$b_3^0 = m_3^{-1}\left(b_3\right)^{-2} v_3 \sum c_{r_1,r_2,...,r_n} 1(b_2)^{v_2}(b_3)^{v_3-1}(b_4)^{v_4}...(b_n)^{v_n},$$
$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$
$$b_n^0 = m_n^{-1}\left(b_n\right)^{n-1} v_n \sum c_{r_1,r_2,...,r_n} 1(b_2)^{v_2}(b_3)^{v_3}(b_4)^{v_4}...(b_{n-1})^{v_{n-1}}...(b_n)^{v_n-1}.$$

It should be noted that the coefficients $b_i^0$ in the differential equation (7) are expressed in terms of “modal constants” $b_i$, which are currently unknown. To establish the relation between them, it will be used compatibility conditions derived from the method of determining the generalized forces of modal oscillations, namely conditions:

$$b_i\frac{\partial P(y,b_2y,...,b_ny)}{\partial y}=\frac{\partial P(y_1,y_2,...,y_n)}{\partial y_i} \tag{8}$$

Substituting the expression (8) into the expression for the potential energy (2), the algebraic relations for determining the unknown quantities $b_i$ take the form:

$$b_i \sum v_i c_{r_1,r_2,\dots,r_n} 1(b_2)^{v_2}(b_3)^{v_3}\dots(b_n)^{v_n} = \sum v_i c_{r_1,r_2,\dots,r_n} 1(b_2)^{v_2}(b_3)^{v_3}\dots(b_{i-1})^{v_{i-1}}(b_i)^{v_i-1}(b_{i+1})^{v_{i+1}}\dots(b_n)^{v_n}. \quad (9)$$

Thus, the issue of the existence of normal modes of oscillations in strongly nonlinear mechanical systems with $n$ degrees of freedom is related to the problem of the existence of real solutions of the nonlinear algebraic equations system (9). This problem has at least $n$ solutions in the case of a homogeneous even function $P(y_1, y_2, \dots, y_n)$. This property of the algebraic equations system (9) is closely related to the phenomenon of "distortions" and the stability of normal modes of oscillations [12], highlighting another fundamental difference between normal modes of oscillations in strongly nonlinear mechanical systems and linear oscillations of the system.

Therefore, the normal modes of oscillation of strongly nonlinear conservative systems, whose potential energy is an even homogeneous function with the exponent $k$ are described using periodic Ateb-functions in the form:

$$y_1 = y = aca\left(v,1,\omega(a)t+\theta\right), \quad (10)$$
$$y_i = y = ab_i ca\left(v,1,\omega(a)t+\theta\right)$$

with $a$, $\theta$ are constants, $\omega(a) = \sqrt{2^{-1}\left(v+1\right)b_1^0}\, a^{\frac{v-1}{2}}$, $b_i$ are determined from the algebraic equations system (9).

The mentioned methodology for constructing solutions of "normal" modes of oscillations can be extended to certain other classes of nonlinear conservative systems, in particular, to systems whose potential energy is determined by the dependence:

$$P = \sum_{i,j=0} c_{ij}\left(y_i - y_j\right)^{v+1}. \quad (11)$$

The differential equations of the conservative system motion can be written in the form:

$$\ddot{y}_i + m_i^{-1}\left(v+1\right)\sum_{j=0} c_{ij}\left(y_i - y_j\right)^{v} = 0, \quad y_0 = 0. \quad (12)$$

Assuming, as in equation (4), the relation between the normal modes of oscillations in the form (5), it is obtained a nonlinear differential equation to find the unknown function $y$:

$$\ddot{y}_i + m_i^{-1}\left(v+1\right)y^{v}\sum_{j=0} c_{ij}\left(b_i - b_j\right)^{v} = 0, \quad i = 2,3,\dots,n, \quad (13)$$

in which the unknown coefficients $b_i$ are determined from the algebraic equations system:

$$b_i \sum_{j=0} c_{ji}\left(1-b_j\right)^{v} = \sum_{j=0} c_{ij}\left(b_i - b_j\right)^{v}. \quad (14)$$

The solution of the differential equation (13) is expressed using periodic Ateb-functions in the form (10), but for the case under consideration, the constants $b_i$ are determined by the algebraic equations system (14).

The described methodology for studying normal modes of oscillations can be generalized to strongly nonlinear systems with small perturbations of both autonomous and non-autonomous types, which are "close" to those considered. Therefore, it can be applied to systems whose motion is described by such differential equations:

$$\ddot{y}_i + m_i^{-1} v_i \sum c_{r_1,r_2,\dots,r_n} y_1^{v_1} y_2^{v_2}\dots y_{i-1}^{v_{i-1}} y_i^{v_i-1} y_{i+1}^{v_{i+1}}\dots y_n^{v_n} = \varepsilon f(y_1,\dots,y_n,\mu t,\varepsilon) \quad (15)$$

or

$$\ddot{y}_i + m_i^{-1}\left(v+1\right)\sum_{j=0} c_{ij}\left(y_i - y_j\right)^{v} = \varepsilon f_i(y_1,\dots,y_n,\mu t,\varepsilon), \quad (16)$$

where the right part of these differential equations (the functions $\varepsilon f_i(y_1,\dots,y_n,\mu t,\varepsilon)$) are analytic and $2\pi$-periodic in $\mu t$.

There was considered oscillations that are close to the normal oscillations of the corresponding unperturbed conservative systems in systems (15) and (16). It is known [13] that forced oscillations of nonlinear systems with one degree of freedom, in the presence of small periodic perturbations, are close to the natural oscillations of the unperturbed system in both resonant and non-resonant cases. It turns out that systems with a finite number of degrees of freedom have similar behavior to conservative systems with one degree of freedom in the modes of normal oscillations. Indeed, according to a general scheme for constructing asymptotic approximations for systems with small perturbations [11], there is:

$$y_i = y_i^0(t) + \varepsilon u_i(t,\varepsilon) \tag{17}$$

with $y_i^0(t)$ is the solution of the unperturbed system, $\varepsilon u_i(t,\varepsilon)$ is the perturbation of the solution caused by the presence of small forces (depending on the right part of the differential equations).

In the case of normal oscillations of the considered systems, there is $y_i^0 = ab_i ca(v+1, 1, \omega(a)t + \theta)$ ($b = 1$, and $b_2$, $b_3, \ldots,$ $b_n$ are constants determined from the corresponding scheme of algebraic equations), $u_i(t,\varepsilon)$ are bounded analytic functions of their arguments, so $y_i \to y_i^0(t)$ for $\varepsilon \to 0$.Thus, substituting the relation (5) for $y_i$ into (15) and (16), it is obtained nonlinear differential equations:

$$\ddot{y}_i + b_i^0 y^{vv} = \varepsilon F_i(y, \mu t, \varepsilon) \tag{18}$$

with $F_i(y, \mu t, \varepsilon) = f_i(y, b_1 y, b_2 y, \ldots, b_n y, \mu t, \varepsilon)$.

It is noticed that the equations for determining the coefficients $b_i$ can also be interpreted as the conditions for the equality of the frequencies of the normal modes of oscillation of the mechanical system. Indeed, the frequency of oscillation of the generalized coordinate for $y_i$ $(i = 2,3,\ldots,n)$ takes the below value, according to formula (14):

$$\omega_i^2(a) = v_i \frac{v+1}{2b_i} a^{v+1} \sum c_{r_1, r_2, \ldots, r_n} b_2^{v_2} b_3^{v_3} \ldots b_{i-1}^{v_{i-1}} b_i^{v_i - 1} b_{i+1}^{v_{i+1}} \ldots b_n^{v_n} . \tag{19}$$

By equating the oscillation frequencies of the generalized coordinates $y_i$ to the frequency of the principal (first) generalized coordinate $y_1$ it is obtained the algebraic equations system for determining the coefficients $b_i$, which coincides with the one given above.

As an example, let's consider the linear oscillations of a system consisting of two masses connected by a system of springs (see Fig. 1).

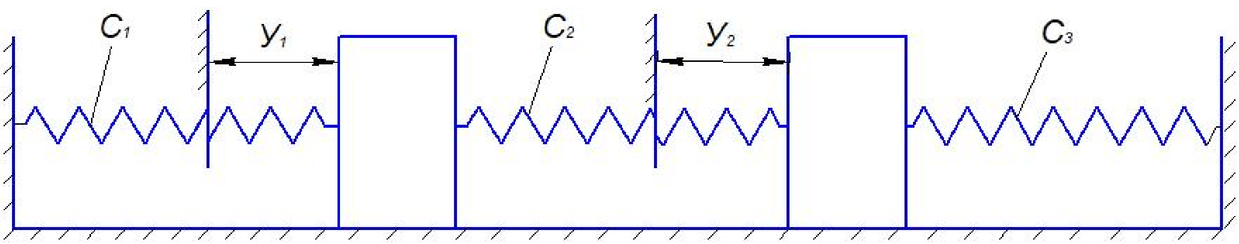


**Fig. 1.** The diagram of a mechanical system with two degrees of freedom

Let's consider that the nonlinear spring characteristics are approximated by dependencies $F_i = c_i(\Delta_i^3 + \alpha_i \Delta_i)$ ($\Delta_i$ is the deformation of the *i*-th spring, $i = 1,2,3$; $c_i$, $\alpha_i$ are constants, moreover $\alpha_i << c_i$), the differential equations of the system motion take the form:

$$\ddot{y}_1 + c_1 y_1^3 + c_2 (y_1 - y_2)^3 = -\varepsilon [k_1 y_1 + k_2 (y_1 - y_2)],$$

$$\ddot{y}_2 + c_3 y_2^3 + c_2 (y_2 - y_1)^3 = -\varepsilon [k_3 y_2 + k_2 (y_2 - y_1)] \tag{20}$$

with $c_i = \varepsilon k_i$.

Let's consider the unperturbed system of equations corresponding to (20), namely:

$$\ddot{y}_1 + c_1 y_1^3 + c_2 \left( y_1 - y_2 \right)^3 = 0 \,,$$

$$\ddot{y}_2 + c_3 y_2^3 + c_2 \left( y_2 - y_1 \right)^3 = 0 \,. \tag{21}$$

By assuming the relation between the normal modes of oscillations $y_1$ and $y_2$ in the form of $y_2 = by_1$, to determine the unknown constant $b$ it is obtained the algebraic equation:

$$b^4 + \left( \frac{c_3}{c_2} - 2 \right) b^3 - \left( \frac{c_1}{c_2} - 2 \right) b - 1 = 0 \,. \tag{22}$$

For $c_1 = c_3 = c$ the real roots of the obtained algebraic equation are equal to:

$$b_1 = 1 \,,\; b_2 = -1 \,,\; b_3 = 1 - \frac{c}{c_2} - \sqrt{\frac{c}{c_2}\left( \frac{c}{4c_2} - 1 \right)} \,,\; b_4 = 1 - \frac{c}{c_2} + \sqrt{\frac{c}{c_2}\left( \frac{c}{4c_2} - 1 \right)} \,. \tag{23}$$

Thus, in the considered conservative system for $a_j = 0 \; \left( k_j = 0, j = 1,2,3,4 \right)$ and $c > 4c_2$, the following forms of normal oscillations are possible:

$$\text{a) } y_1 = aca\left( 3,1, \sqrt{c} \cdot at + \theta \right),$$

$$y_2 = aca\left( 3,1, \sqrt{c} \cdot at + \theta \right);$$

$$\text{b) } y_1 = aca\left( 3,1, \sqrt{2c_2 + c} \cdot at + \theta \right),$$

$$y_2 = -aca\left( 3,1, \sqrt{2c_2 + c} \cdot at + \theta \right);$$

$$\text{c) } y_1 = aca\left( 3,1, \sqrt{c + c_2\left( \frac{c}{2c_2} + \sqrt{\frac{c^2 - 4cc_2}{4c_2^2}} \right)^3} \cdot at + \theta \right),$$

$$y_2 = \left( 1 - \frac{c}{2c_2} - \sqrt{\frac{c^2 - 4cc_2}{4c_2^2}} \right) aca\left( 3,1, \sqrt{c + c_2\left( \frac{c}{2c_2} + \sqrt{\frac{c^2 - 4cc_2}{4c_2^2}} \right)^3} \cdot at + \theta \right); \tag{24}$$

$$\text{d) } y_1 = aca\left( 3,1, \sqrt{c + c_2\left( \frac{c}{2c_2} - \sqrt{\frac{c^2 - 4cc_2}{4c_2^2}} \right)^3} \cdot at + \theta \right),$$

$$y_2 = \left( 1 - \frac{c}{2c_2} + \sqrt{\frac{c^2 - 4cc_2}{4c_2^2}} \right) aca\left( 3,1, \sqrt{c + c_2\left( \frac{c}{2c_2} - \sqrt{\frac{c^2 - 4cc_2}{4c_2^2}} \right)^3} \cdot at + \theta \right).$$

From the obtained results, it follows that:

– for the case *a* the normal modes of oscillations $y_1$ and $y_2$ occur in the same phase, while for the case *b* they occur in the opposite phases;

– normal oscillations in the forms *c* and *d* occur if the spring stiffnesses are related by the dependency $c > 4c_2$.

Having determined the normal modes of oscillations of the unperturbed system (21), it proceeded to consider the first-order perturbed equations. According to the general perturbation theory for nonlinear oscillatory systems, the solution of the equations (21) can also be considered as the relation (24), but in this case, the parameters $a$ and $\theta$ will be functions of time. To find one, it is obtained the differential equations system:

$$\dot{a} = \frac{\varepsilon a}{2\pi\omega(a)} \int_0^{2\pi} k(1+b)ca(3,1,l\psi)sa(1,3,l\psi)d\psi = 0,$$

$$\dot{\theta} = \frac{\varepsilon}{2\pi\omega(a)} \int_0^{2\pi} k(1+b)ca^2(3,1,l\psi)d\psi = \frac{\varepsilon 0.4571}{\omega(a)} k(1+b) \quad (25)$$

with $k = k_1 = k_2$, $l = \frac{\Gamma(0.25)}{\sqrt{\pi}\Gamma(0.75)} = 1.6692$.

As expected, the amplitudes of the normal modes of oscillations of the considered system in the first-order approximation of the asymptotic expansion remain constant because the system is conservative. As for the frequencies of the perturbed oscillations $\Omega_s$, $s = 1,2,3,4$ they depend on the amplitude and are determined by the formula:

$$\Omega_s = \omega(a) + \frac{\varepsilon 0.4571}{\omega(a)} k(1+b) \quad (26)$$

with $\omega(a) = \sqrt{\frac{\pi}{2}\left[c + c_2(1+b_s)^3\right]} a$, then $b_s$ are determined according to (23).

Below, in Fig. 2, the dependence of the period of oscillation normal modes on the amplitude is depicted.

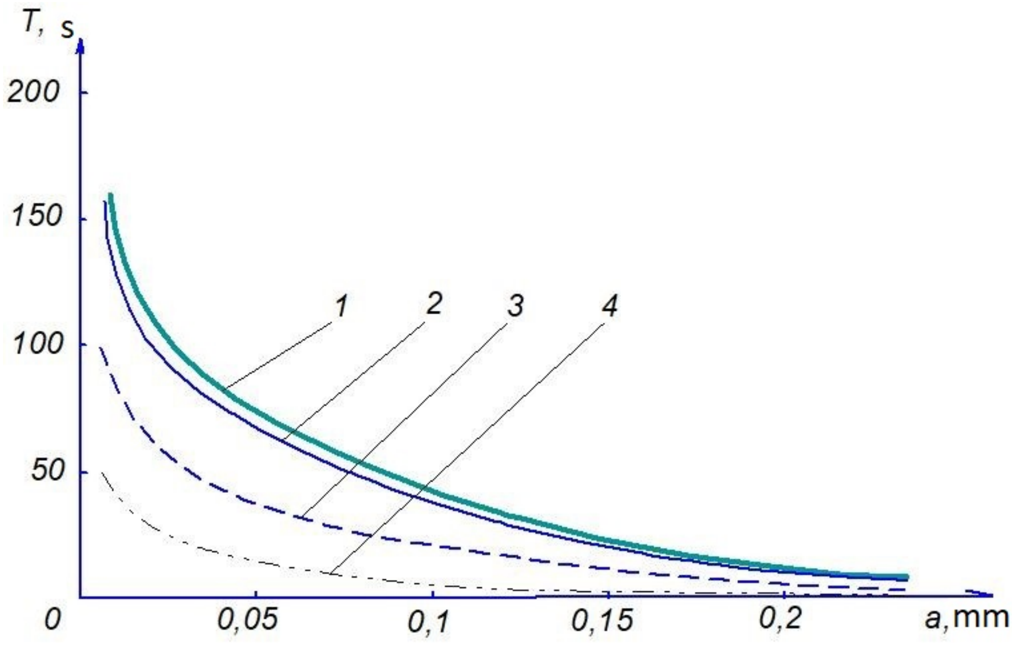


**Fig. 2.** Graphs illustrating the dependence of the periods of oscillation normal modes on the amplitude: 1 – a; 2 – b; 3 – c; 4 – d

## Conclusions

From the obtained graphs, it can be observed that:

– the period of all oscillation normal modes of the investigated conservative system decreases with increasing amplitude;

– the period of oscillations is highest for the case of in-phase motion of the bodies for the same amplitude values.

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