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JACOBSTHAL RECURRENT NUMBERS AS A PLATFORM FOR TRANSFORMATIONS K·Q±1

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Abstract. The paper investigates the role of Jacobsthal recurrent numbers in forming statistical patterns within the model of the natural number hypothesis q ϵ ℕ in the general problem of the form κ·q±1, where κ=1,3,5,…. A novel model is proposed for structuring the set of natural numbers as sequences of the form $\theta\cdot 2^n$, where the parameter θ takes odd values 1,3,5,…, and n is a natural number starting from zero. A branching and merging diagram of such sequences has been developed, describing their evolution towards a general stopping time tst, where tst→∞. The properties of these structures are investigated, particularly their relationship with the dynamics of the Collatz conjecture. Based on the proposed model, the formation of number sequences with the same length in the Collatz conjecture CSq has been identified for the first time. The obtained results can be used for further analysis of arithmetic transformations and properties of natural numbers in the context of number theory.

Keywords: recurrence number, histogram, Jacobsthal number, total stopping time, Collatz conjecture, probability distribution function.

Introduction

In mathematics, the Collatz problem [1,2] is known, in which a sequence of natural numbers q ϵ ℕ is generated. Members of such a sequence are calculated according to the rule: let the number *q* be even, then the next member of the sequence is equal to *q*/2, otherwise if q is odd (q_{odd}), then the next member of the sequence is calculated as $C^{+}_{3q} = 3q + 1$.The sequence for which this rule is fulfilled will be called the Collatz sequence, and its graph will be called the Collatz trajectory. The Collatz sequence (CS_q) is calculated by the algorithm

$$C_{3q+1} = \quad if \quad q \equiv 0 \mod 2 \quad then \quad \frac{q}{2} \quad else \quad C_{3q+1} = 3q + 1 \tag{1}$$

Therefore, the last member of the CS_q is the unit by which the periodic cycle is formed $cycle_{3q,1\leftrightarrow 1} = 1 \to 4 \to 2 \to 1 \to 4 \to 2 \to 1 \to \dots$.

Main material presentation

Etude I: Bařina’s 5(7)x±1 Conjectures. Similar (1) sequences of David Bařina's 7x+1 and mod 8 conjecture [3-5] are known, which are calculated as:

$$C_{7q} = \begin{cases} 7q-1 & if \quad x \equiv 1 \pmod 4 \\ 7q+1 & if \quad x \equiv -1 \pmod 4 \; [3,4] \\ q/2 & if \quad x \equiv 0 \pmod 2 \end{cases} \tag{2}$$

$$C_{5q} = \begin{cases} 5q-1 & if \quad x \equiv 1 \pmod 8 \quad (a) \\ 5q+1 & if \quad x \equiv -1 \pmod 8 \quad (b), \\ \lfloor q/2 \rfloor & otherwise \quad (c) \end{cases} \tag{1.1}$$

which also leads to unity, where the functions $C^{\pm}_{5(7)q}$ of the conjecture of an odd number q_{odd} are given by the type of remainders from division q_{odd} by 4,8. However, the CS_q cannot have two adjacent odd numbers, while calculated by an algorithm (1.1), the sequence:

1111, 5556, 2778, 1389, 694. 347, 173, 86, 43, 21, 10, 5, 2, 1,

starting number = 1111 (1.2)

the number of iterations taken to reach 1 is 13 has fragments with two adjacent odd q_{odd} (red) numbers.

The purpose of this work is to show that the regularities of conjectures of numbers $q \in \mathbb{N}$ according to the analogy (1)

$$C^{\pm}_{5(7)q} = \quad if \quad q \equiv 0 \mod 2 \quad then \quad \frac{q}{2} \quad else \quad 5(7)q \pm 1 \;, \tag{1.3}$$

algorithm, fundamentally different from *David Bařina's 5(7)x+1 conjecture* [3-5]. We will investigate the formulated problem from the point of view of parameterized θ recurrent Jacobsthal numbers [6-7].

$$m(p)_{\kappa,\theta,n} = \frac{\theta \cdot 2^n - (+)1}{\kappa} \quad , n \in \mathbb{N} \cup \{0\}, \mathbb{N}=\mathbb{N}_{odd} \cup \mathbb{N}_{even}, \; \theta, \kappa \in \mathbb{N}_{odd}, \tag{1.4}$$

which form the nodes of the binary sequence $\{\theta \cdot 2^n\}_{n=0}^{n=\infty}$ and we will examine their correlations with the regularities of 4 and $\mod 8$ residue formation (1.1). For $\kappa = 7$, numbers $m(p)_{7,\theta,n} = \dfrac{\theta \cdot 2^n - (+)1}{7}$ are given in Table 1 for $\theta = 1 \div 17$:

Table 1

n	0	1	2	3	4	5	6	7	8	9	10	
θ=1	0 -			1 -			9 -			73 -		$m_{7,1,n+1}=8\cdot m_{7,1,n}+1$
θ=3		- 1			- 7			- 55			- 439	$p_{7,3,n+1}=8\cdot p_{7,3,n}-1$
θ=5			- 3			- 23			- 183			$p_{7,5,n+1}=8\cdot p_{7,5,n}-1$
θ=9			5 -			41 -			329 -			$m_{7,9,n+1}=8\cdot m_{7,9,n}+1$
θ=11		3 -			25 -			201 -			1609 -	$m_{7,11,n+1}=8\cdot m_{7,11,n}+1$
θ=13	2			- 15			119			- 951		$p_{7,13,n+1}=8\cdot p_{7,13,n}-1$
θ=15	2			17 -			137			1097 -		$m_{7,15,n+1}=8\cdot m_{7,15,n}+1$
θ=17		- 5			- 39			- 311			- 2487	$p_{7,17,n+1}=8\cdot p_{7,17,n}-1$

If θ is divisible by 7 ($\theta = \theta_7$ = 7, 21, 27, ...), then the numbers ${}_7J^{\pm}_{\theta,n}$ are fractional and not listed in Table 1.

The numbers $m_{7,1,n}$ of the first θ =1 row of table 1 form nodes of the sequence $\left\{1\cdot 2^n\right\}_{n=0}^{n=\infty}$. The numbers $m(p)_{7,3,n}$ of the second line form nodes of the sequence $3\cdot 2^n$ and so on. In the last column of table 1, there are formulas that connect two adjacent numbers $m(p)_{7,\theta,n}$ for each row

$$m(p)_{7,\theta,n+1} = 8\cdot m(p)_{7,\theta,n+1} + (-)1 \tag{1.5a}$$

Formulas (1.5a) are similar to formulas

$$m(p)_{3,\theta,n+1} = 4\cdot m(p)_{3,\theta,n+1} + (-)1 \tag{1.5b}$$

in problem $C^{\pm}_{3q}$ [8]. Formulas of type (1.5) represent an individual marker of conjecture functions $C^{\pm}_{7(3)q}$. Indeed, the difference between two adjacent numbers (1.5a) is equal:

$$m(p)_{7,\theta,n+1} - \cdot m(p)_{7,\theta,n} = 7q \pm 1 \quad \text{and} \quad m(p)_{3,\theta,n+1} - \cdot m(p)_{3,\theta,n} = 3q + (-)1 \tag{1.6}$$

and equal the functions $C^{\pm}_{7(3)q}$.

The sequence $C^{\pm}_{7q}$ merges with the sequence $\left\{1\cdot 2^n\right\}_{n=0}^{n=\infty}$ through those nodes $m(p)_{7,1,n}$ for which the equality holds

$$1\cdot 2^n = 7m(p)_{7,1,n} + (-)1. \tag{1.7}$$

As can be seen from the first line of Table 1.1, the numbers $m_{7,1,n}$ are integers. Therefore, the sequence of the Collatz conjecture $C^{\pm}_{7q}$ can reach the unit element $1\cdot 2^0$ of the sequence $\left\{1\cdot 2^n\right\}_{n=0}^{n=\infty}$ in a finite number of iterations. The $p_{7,1,n}$ numbers are fractional, so for the function C^{-}_{7q} the nodes of the sequence $\left\{1\cdot 2^n\right\}_{n=0}^{n=\infty}$ do not form. Thus, the sequence of C^{-}_{7q} conjecture is isolated from the sequence $\left\{1\cdot 2^n\right\}_{n=0}^{n=\infty}$ and cannot reach the unit element for a finite number of iterations.

Now consider the numbers $m(p)_{5,\theta,n}$

$$m(p)_{5,\theta,n} = \frac{\theta\cdot 2^n - (+)1}{5}, \tag{1.8}$$

the values of which are given in Table 1.2 for parameter θ from $\theta = 1$ to $\theta = 13$.

The numbers (1.8) for which θ are multiples of five ($\theta = \theta_5$ =5,15,25,…) are fractional, so they are not listed in Table 2.

Two adjacent numbers (1.8) in each row are connected by a ratio

$$m(p)_{5,\theta,n+1} = 16\cdot m(p)_{5,\theta,n} + (-)3, \tag{1.9}$$

and the function marker $C^{\pm}_{5q}$ is equal to:

$$m(p)_{5,\theta,n+1} - \cdot m(p)_{5,\theta,n} = 3(5q + (-)1) = 3C^{\pm}_{5q}. \tag{1.10}$$

However, in contrast to the numbers $m(p)_{7,1,n}$, for both numbers $m(p)_{5,1,n}$ in the first row of Table 2, equalities are fulfilled

$$1\cdot 2^n = 5\cdot m(p)_{5,\theta,n} + (-)1. \tag{1.11}$$

Thus, the trajectories of both $C^{\pm}_{5q}$ conjectures through the nodes $m(p)_{5,1,n}$ of the sequence $\left\{1\cdot 2^n\right\}_{n=0}^{n=\infty}$ can reach unity in a finite number of iterations.

Therefore, the sequences of *David Bařina's 5(7)x+1 conjecture* is not Collatz sequences, but represent a separate class of recurrent sequences that also converge to unity in a finite number of iterations.

Table 2

n	**0**	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**10**	
θ=1			1				13				205	$p_{5,1,n+1}=16\cdot p_{5,1,n}-3$
	0				3				51			$m_{5,1,n+1}=16\cdot m_{5,1,n}+3$
θ=3				5				77				$p_{5,3,n+1}=16\cdot p_{5,3,n}-3$
		1				9				307		$m_{5,3,n+1}=16\cdot m_{5,3,n}+3$
θ=7		3				45				717		$p_{5,7,n+1}=16\cdot p_{5,7,n}-3$
				11				179				$m_{5,7,n+1}=16\cdot m_{5,7,n}+3$
θ=9	2				29				461			$p_{5,9,n+1}=16\cdot p_{5,9,n}-3$
			7				115				1843	$m_{5,9,n+1}=16\cdot m_{5,9,n}+3$
θ=11			9				141				2253	$p_{5,11,n+1}=16\cdot p_{5,11,n}-3$
	2				35				563			$m_{5,11,n+1}=16\cdot m_{5,11,n}+3$
θ=13				21				333				$p_{5,13,n+1}=16\cdot p_{5,13,n}-3$
		5				83				1331		$m_{5,13,n+1}=16\cdot m_{5,13,n}+3$

To confirm this conclusion, let's express relation (1.5a) in the form

$$m(p)_{(3)7,\theta,n+1} = 2\bmod 2(4)\cdot m(p)_{(3)7,\theta,n} + (-)1, \qquad (1.12)$$

connect residues $\bmod 2$ and $\bmod 4$ with nodes $m(p)_{\kappa,\theta,n}$. Then the regularities of the formation of residues $\bmod 2$ and $\bmod 4$ for the first eight functions $C^{\pm}_{3\div17,q}$ of conjectures of numbers in nodes with Jacobsthal numbers will have the form:

mod 2	+1	+1	+1	+1	+1	+1	+1	+1	+1	+1	+1	+1	+1	+1	3x+1
q_{odd}	1	3	5	7	9	11	13	15	17	19	21	23	25	27	
mod 2	-1	-1	-1	-1	-1	-1	-1	-1	-1	-1	-1	-1	-1	-1	3x-1
mod 4			+1		+1		+1		+1		+1		+1		7(9)x-1
q_{odd}	1	3	5	7	9	11	13	15	17	19	21	23	25	27	
mod 4		-1		-1		-1		-1		-1		-1		-1	7(9)x+1
mod 8					+1				+1				+1		5(15,17)x-1
q_{odd}	1	3	5	7	9	11	13	15	17	19	21	23	25	27	
mod 8				-1				-1				-1			5(15,17)x+1
mod 16									+1						11x+1
q_{odd}	1	3	5	7	9	11	13	15	17	19	21	23	25	27	
mod 16								-1							11x-1
mod 32															13x+1
q_{odd}	1	3	5	7	9	11	13	15	17	19	21	23	25	27	
mod 32															13x-1

We see that when in function $C_{\kappa q\pm1}$ the parameter κ>3, in the set of numbers q_{odd}, intervals of values (yellow) :

q_{odd} =1, [7(9)x +/-1];
1÷5, 13,15, 19,21,…, [5(15,17)x+/-1];
1÷13, 19÷31,…, [11x+/-1]; (1.13)
1÷31,…, [13x+/-1];…,

are formed, which require the formulation of additional conditions, as of type (1.1c) that are not related to the Collatz problem. Therefore, there are no periodic cycles of type $cycle_{3q,1\leftrightarrow1}$ in cases (1.13). However, q_{odd} conjectures are carried out according to the same rules of type $\kappa x + (-)1$ (gray) and (1.1c) (yellow). Diagram (1.13) also confirms the shows the uniform distribution of residues $\bmod_{\kappa}$.

Etude II: Fractal Structures Using Jacobsthal Numbers. It is known that the classical Collatz function (1) is linear and separately describes the transformation of even $q_{even} \in N_{even}$ and odd

$q_{odd} \in \mathrm{N}_{odd}$ natural numbers $q \in \mathrm{N} = \mathrm{N}_{odd} \cup \mathrm{N}_{even}$. Therefore, to construct Collatz fractals, the authors [10-12] used the abbreviated [13] form

$$C_{3,q}^{+} = (3q+1)/2\,, \tag{2.1}$$

and got the function

$$C_{3,q_{N+1}}^{+} = q_{N+1} = \frac{7q_N + 2 + (-1)^{q_N}(5q_N + 2)}{4} \tag{2.2}$$

In this work, a general class of functions $\mathrm{K}_{\kappa,q_{N+1}}^{\pm}$ conjectures are constructed [7]

$$C_{\kappa,q}^{\pm} = \begin{cases} C_{\kappa,q/2}^{\pm} = \dfrac{q}{2} & if \;\; q \;\; is \;\; even, \\ C_{\kappa q \pm 1}^{\pm} = \kappa q \pm 1 & if \;\; q \;\; is \;\; odd, \end{cases} \quad \kappa = 1,3,5,... \in \mathrm{N}_{odd} \tag{2.3}$$

Consider the $\mathrm{K}_{\kappa,q_{N+1}}^{\pm}$ function in the form

$$\mathrm{K}_{\kappa,q_{N+1}}^{\pm} = \alpha_\kappa(q_N)\cdot \mathrm{K}_{\kappa,q_N}^{\pm}/4 + \beta_\kappa(q_N)\mathrm{K}_{\kappa,q_N}^{\pm} \tag{2.4}$$

Multipliers $\beta_\kappa(q_N)$ and $\alpha_\kappa(q_N)$ satisfy the conditions

$$\beta_\kappa(q_N) = 0 \quad if \quad q = q_{even} \quad \text{and} \quad \alpha_\kappa(q_N) = 0 \quad if \quad q_N = q_{odd}\,, \tag{2.5}$$

therefore, let's represent them as:

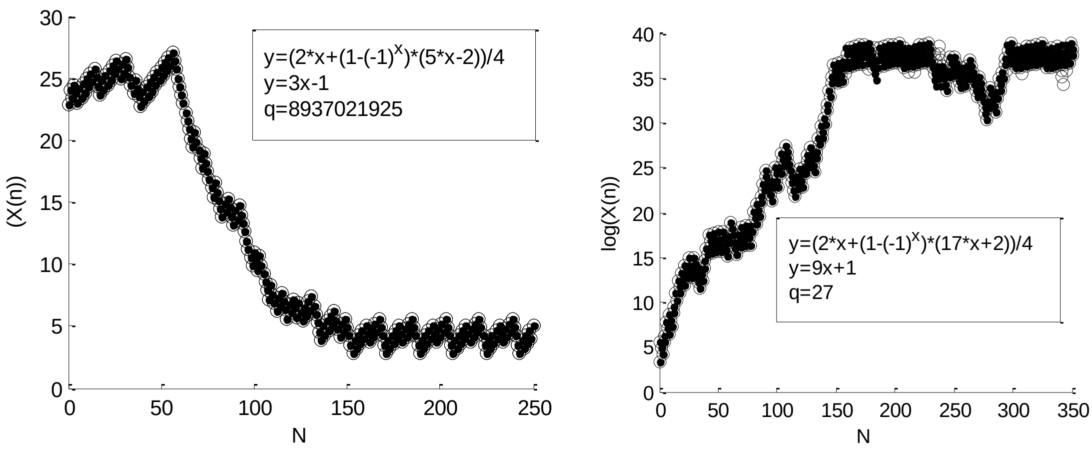
$$\begin{cases} \alpha_\kappa(q_N) = \dfrac{1+(-1)^{q_N}}{2} \Rightarrow \begin{cases} if \quad q_N = q_{even} \quad then \quad \alpha_\kappa(q_{even}) = 1, \\ if \quad q_N = q_{odd} \quad then \quad \alpha_\kappa(q_{odd}) = 0. \end{cases} \\ \beta_\kappa(q_N) = \dfrac{1-(-1)^{q_N}}{2} \Rightarrow \begin{cases} if \quad q_N = q_{odd} \quad then \quad \beta_\kappa(q_{odd}) = 1, \\ if \quad q_N = q_{even} \quad then \quad \beta_\kappa(q_{even}) = 0. \end{cases} \end{cases} \tag{2.6}$$

Expressions (2.6) do not depend on the κ parameter, so by substituting (2.6) into (2.4) we obtain:

$$\mathrm{K}_{\kappa,q_{N+1}}^{\pm} = \frac{2\mathrm{K}_{\kappa,q_N}^{\pm} + [1-(-1)^{\mathrm{K}_{\kappa,q}^{\pm}}]((2\kappa-1)\cdot \mathrm{K}_{\kappa,q_N}^{\pm} \pm 2)}{4} \tag{2.7}$$

If the number is even, then the value of the converted number at N iterations is calculated as $\mathrm{K}_{\kappa,q_{N+1}}^{\pm} = \dfrac{\mathrm{K}_{\kappa,q_N}^{\pm}}{2}$. If it is odd, the value of the converted number at N iterations is calculated as

$$\mathrm{K}_{\kappa,q_{N+1}}^{\pm} = \frac{\mathrm{K}_{\kappa,q_N}^{\pm} + ((2\kappa-1)\cdot \mathrm{K}_{\kappa,q_N}^{\pm} \pm 2)}{2} = = \kappa \cdot \mathrm{K}_{\kappa,q_N}^{\pm} \pm 1. \tag{2.8}$$



$\kappa = 3$ $\qquad\qquad$ $\kappa = 5$

Fig. 1. The graphs of the sequences of Collatz

Plotted in Fig.1, the graphs of the sequences of Collatz $C_{3,q}^{-}$, *and* $C_{9,q}^{+}$, and calculated by formula (2.7) show that the results of the calculations by both approaches coincide with each other. The graphs are plotted in a semi-logarithmic scale using the functions recorded in the insets. The parameters of the number conversion model are chosen arbitrarily.

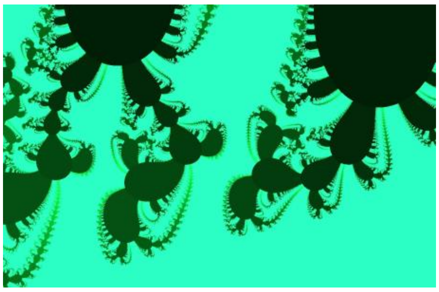
In (2.7), the expression $(-1)^{q_N}$ (for even numbers is equal + (plus) and is equal – (minus) for odd number) plays the role of a sign function. Therefore, replacing $(-1)^{q}=\cos(\pi q)$, $1-\cos(\pi q)=2\sin^2\left(\frac{\pi q}{2}\right)$ in (2.7), we get formula (2.6) in the form:

$$\mathrm{K}^{\pm}_{\kappa,q_{N+1}}=\frac{\mathrm{K}^{\pm}_{\kappa,q_N}+\sin^2\left(\mathrm{K}^{\pm}_{\kappa,q_N}/2\right)[(2\kappa-1)\cdot\mathrm{K}^{\pm}_{\kappa,q_N}\pm1]}{2} \tag{2.9}$$

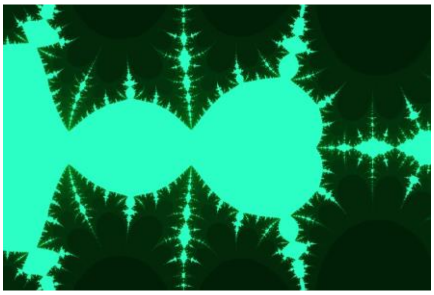
In Fig.1 fractals constructed by three functions $\mathrm{K}^{\pm}_{\kappa,q_{N+1}}$ (2.9) with the parameter $\kappa=1$, for which the Collatz function (2.3) of the transformation of odd numbers has the form $C^{+}_{1,q_{N+1}}=1q_N+1$ [7,17-18]:

$$\begin{aligned}
a:&\quad \mathrm{K}^{+}_{1,q_{N+1}}=\frac{2\mathrm{K}^{+}_{1,q_N}+\left(1-(-1)^{\mathrm{K}^{+}_{1,q_N}}\right)\left[\mathrm{K}^{+}_{1,q_N}+2\right]}{4}\\
b:&\quad \mathrm{K}^{+}_{1,q_{N+1}}=\frac{\mathrm{K}^{+}_{1,q_N}+\left(\sin\frac{\pi}{2}\mathrm{K}^{+}_{1,q_N}\right)^2\left[\mathrm{K}^{+}_{1,q_N}+1\right]}{2}\\
c:&\quad \mathrm{K}^{+}_{1,q_{N+1}}=\frac{2\mathrm{K}^{+}_{1,q_N}+\left(1-[\exp(-i\cdot\pi\cdot\mathrm{K}^{+}_{1,q_N})]\right)\left[\mathrm{K}^{+}_{1,q_N}+2\right]}{4}
\end{aligned} \tag{2.10}$$

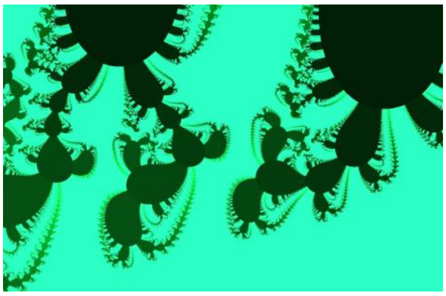
In the case of Fig. 2a, the function (2a) with the $(-1)^{q_N}$ sign switch was used. In the case of Fig. 2b, the function (2.10b) with the $\cos(\pi\cdot\mathrm{K}_{1,q_N})$ sign switch was used. In the case of Fig. 2c, the function (2.10c) with the $\exp(i\cdot\pi\cdot\mathrm{K}_{1,q_N})$ sign switch was used.



a: $\mathrm{K}^{+}_{1,q}$ (2.8)



c: $\mathrm{K}^{+}_{1,q}$ (2.10)



b: $\mathrm{K}^{+}_{1,q}$ (2.9)

Fig. 2 The fractals

We see that the fractals in Fig. 2a and Fig. 2b differ from each other. In the case of function (2.7), we have a fractal of the Julio set type, while the trigonometric function is multivalued, so we have a fractal of the Mandelbrot set type [14-16].

$N := 0, 1 .. 20$ $\qquad K1_0 := 7 \quad K2_0 := 7 \quad K3_0 := 7$

$$K1_{N+1} := \frac{\left[2 \cdot K1_N + \left(\sin\left(\pi \cdot \frac{K1_N}{2}\right)\right)^2 \cdot 2 \cdot (5 \cdot K1_N + 2)\right]}{4} \qquad K2_{N+1} := \frac{\left[2 \cdot K2_N + \left[1 - (-1)^{K2_N}\right] \cdot (5 \cdot K2_N + 2)\right]}{4}$$

$$K3_{N+1} := \mathrm{Re}\left[\frac{2 \cdot K3_N + \left(1 - e^{-i \cdot \pi \cdot K3_N}\right) \cdot (5 \cdot K3_N + 2)}{4}\right] \qquad K\_3_{N+1} := \mathrm{Im}\left[\frac{2 \cdot K3_N + \left(1 - e^{-i \cdot \pi \cdot K3_N}\right) \cdot (5 \cdot K3_N + 2)}{4}\right]$$

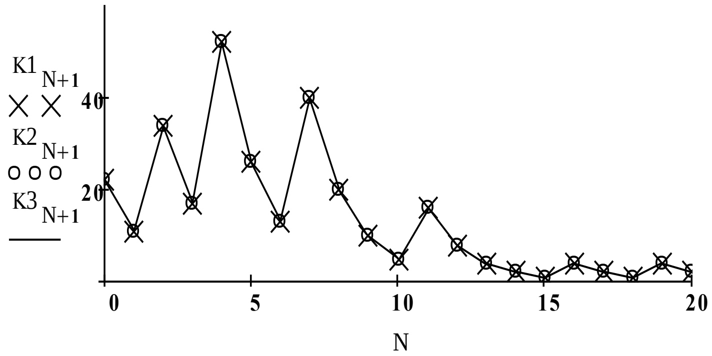


Fig. 3. Statistical Modelling of Collatz Conjectures

Etude III: Statistical Modelling of Collatz Conjectures. In a recently published work [21], the author, investigating the regularities of transforming even numbers q_{even} in the Collatz problem, concluded that a single division $\nu = 1$ resulting in an even number of odd value is implemented with a probability of 50:50, and for a larger number of divisions $\nu_{q/2} \rangle 1$, the ratio is 3:1. We will show that the described transformation regularities of $q_{even}/2$ are strictly deterministic and are determined by the regularities of transformations of natural numbers in nodes with recurrent Jacobsthal numbers $J^{\pm}_{a,\theta,n} = \frac{\theta \cdot 2^n \pm (-1)^n}{a}$ [7], independently of the Collatz problem

$$C^{\pm}_{a,q} = \begin{cases} q/2 & if \;\; q \;\; is \;\; q_{\text{even}} \\ aq \pm 1 & if \;\; q \;\; is \;\; q_{odd} \end{cases}, \; a = 1,3,5,... \tag{3.1}$$

The type of transformation function for odd numbers q_{odd} will be discussed. Let us recall from [7] that the Jacobsthal problem relates to the transformation of natural numbers in the direction of increasing $n \to +\infty$ powers of sequences $\{\theta \cdot 2^n\}_{n=0}^{\infty}$, while the Collatz problem involves their decrease in the direction of $n \to +\infty$.

Consider the set $\mathrm{N} = 1,2,3,4,5,..., q,...\infty$ of natural numbers and transform it into a set of parameterized binary-based sequences $\{\theta \cdot 2^n\}_{n=0}^{\infty} (n \in \mathrm{N} \cup \{0\})$ for the natural numbers $(\mathrm{N} = \mathrm{N}_{odd} \cup \mathrm{N}_{even})$:

$$\begin{aligned} \mathrm{N} = & 1 \cdot 2^0, 1 \cdot 2^1, 1 \cdot 2^2, 1 \cdot 2^3, 1 \cdot 2^4, \ldots, 1 \cdot 2^n, \ldots, \\ & 3 \cdot 2^0, 3 \cdot 2^1, 3 \cdot 2^2, 3 \cdot 2^3, 3 \cdot 2^4, \ldots, 3 \cdot 2^n, \ldots, \\ & 5 \cdot 2^0, 5 \cdot 2^1, 5 \cdot 2^2, 5 \cdot 2^3, 5 \cdot 2^4, \ldots, 5 \cdot 2^n, \ldots, \\ & 7 \cdot 2^0, 7 \cdot 2^1, 7 \cdot 2^2, 7 \cdot 2^3, 7 \cdot 2^4, \ldots, 7 \cdot 2^n, \ldots \\ = & \left\{ \{1 \cdot 2^n\}_{n=0}^{\infty}, \{3 \cdot 2^n\}_{n=0}^{\infty}, \{5 \cdot 2^n\}_{n=0}^{\infty}, \{7 \cdot 2^n\}_{n=0}^{\infty}, \ldots, \{\theta \cdot 2^n\}_{n=0}^{\infty}, \ldots \right\}. \end{aligned} \tag{3.2}$$

The illustration of structuring (3.2) is shown in Fig. 4. Here, the sequences $\{\theta \cdot 2^n\}_{n=0}^{\infty}$ are formatted according to the same powers of doubling. Therefore, their parameter θ is the first $\theta \cdot 2^0$ element, which, in the direction of $n \to +\infty$ doubling, forms even members $q_{even} = \theta \cdot 2^n$. Thus, in the structuring model of the set in Fig. 4, the power of n determines the value of $\nu_{q/2}$.

$$n = \nu_{q/2}, \tag{3.3}$$

conjectures in (3.2) of an even number, until it reaches odd value $\theta \cdot 2^0$ in the direction according to the algorithm:

$$\underbrace{((q_e/2)/2)/2 \to (((q_e/2)/2)/2)/2 \to \ldots \to q_{odd}}_{\nu_{q/2}}, \tag{3.4}$$

regardless of the type of function $C^{\pm}_{\kappa,q}$ for the transformation of odd numbers.

$n=\nu_{q/2}$	$1\cdot 2^n$	$3\cdot 2^n$	$5\cdot 2^n$	$7\cdot 2^n$	$9\cdot 2^n$	$11\cdot 2^n$	$13\cdot 2^n$	…	$\theta\cdot 2^n$
0	$1\cdot 2^0$	$3\cdot 2^0$	$5\cdot 2^0$	$7\cdot 2^0$	$9\cdot 2^0$	$11\cdot 2^0$	$13\cdot 2^0$	…	$\theta\cdot 2^0$
1	$1\cdot 2^1$	$3\cdot 2^1$	$5\cdot 2^1$	$7\cdot 2^1$	$9\cdot 2^1$	$11\cdot 2^1$	$13\cdot 2^1$	…	$\theta\cdot 2^1$
2	$1\cdot 2^2$	$3\cdot 2^2$	$5\cdot 2^2$	$7\cdot 2^2$	$9\cdot 2^2$	$11\cdot 2^2$	$13\cdot 2^2$	…	$\theta\cdot 2^2$
3	$1\cdot 2^3$	$3\cdot 2^3$	$5\cdot 2^3$	$7\cdot 2^3$	$9\cdot 2^3$	$11\cdot 2^3$	$13\cdot 2^3$	…	$\theta\cdot 2^3$
4	$1\cdot 2^4$	$3\cdot 2^4$	$5\cdot 2^4$	$7\cdot 2^4$	$9\cdot 2^4$	$11\cdot 2^4$	$13\cdot 2^4$	…	$\theta\cdot 2^4$
5	$1\cdot 2^5$	$3\cdot 2^5$	$5\cdot 2^5$	$7\cdot 2^5$	$9\cdot 2^5$	$11\cdot 2^5$	$13\cdot 2^5$	…	$\theta\cdot 2^5$
6	$1\cdot 2^6$	$3\cdot 2^6$	$5\cdot 2^6$	$7\cdot 2^6$	$9\cdot 2^6$	$11\cdot 2^6$	$13\cdot 2^6$	…	$\theta\cdot 2^6$
7	$1\cdot 2^7$	$3\cdot 2^7$	$5\cdot 2^7$	$7\cdot 2^7$	$9\cdot 2^7$	$11\cdot 2^7$	$13\cdot 2^7$	…	$\theta\cdot 2^7$
8	$1\cdot 2^8$	$3\cdot 2^8$	$5\cdot 2^8$	$7\cdot 2^8$	$9\cdot 2^8$	$11\cdot 2^8$	$13\cdot 2^8$	…	$\theta\cdot 2^8$
9	$1\cdot 2^9$	$3\cdot 2^9$	$5\cdot 2^9$	$7\cdot 2^9$	$9\cdot 2^9$	$11\cdot 2^9$	$13\cdot 2^9$	…	$\theta\cdot 2^9$
10	$1\cdot 2^{10}$	$3\cdot 2^{10}$	$5\cdot 2^{10}$	$7\cdot 2^{10}$	$9\cdot 2^{10}$	$11\cdot 2^{10}$	$13\cdot 2^{10}$	…	$\theta\cdot 2^{10}$
11	$1\cdot 2^{11}$	$3\cdot 2^{11}$	$5\cdot 2^{11}$	$7\cdot 2^{11}$	$9\cdot 2^{11}$	$11\cdot 2^{11}$	$13\cdot 2^{11}$	…	$\theta\cdot 2^{11}$
12	$1\cdot 2^{12}$	$3\cdot 2^{12}$	$5\cdot 2^{12}$	$7\cdot 2^{12}$	$9\cdot 2^{12}$	$11\cdot 2^{12}$	$13\cdot 2^{12}$	…	$\theta\cdot 2^{12}$
…	…	…	…	…	…	…	…	…	…

Fig. 4 Illustration of the grouping of the form (3.2)

Representing the set N as sequences of $\{\theta \cdot 2^n\}_{n=0}^{\infty}$ allows implementing one of the conditions for its points in the $n \to \infty$ direction

$$\frac{\theta \cdot 2^n - 1}{\kappa} = \theta_{new} \quad \text{(a)}, \qquad \frac{\theta \cdot 2^n + 1}{\kappa} = \theta_{new} \quad \text{(b)}, \tag{3.5}$$

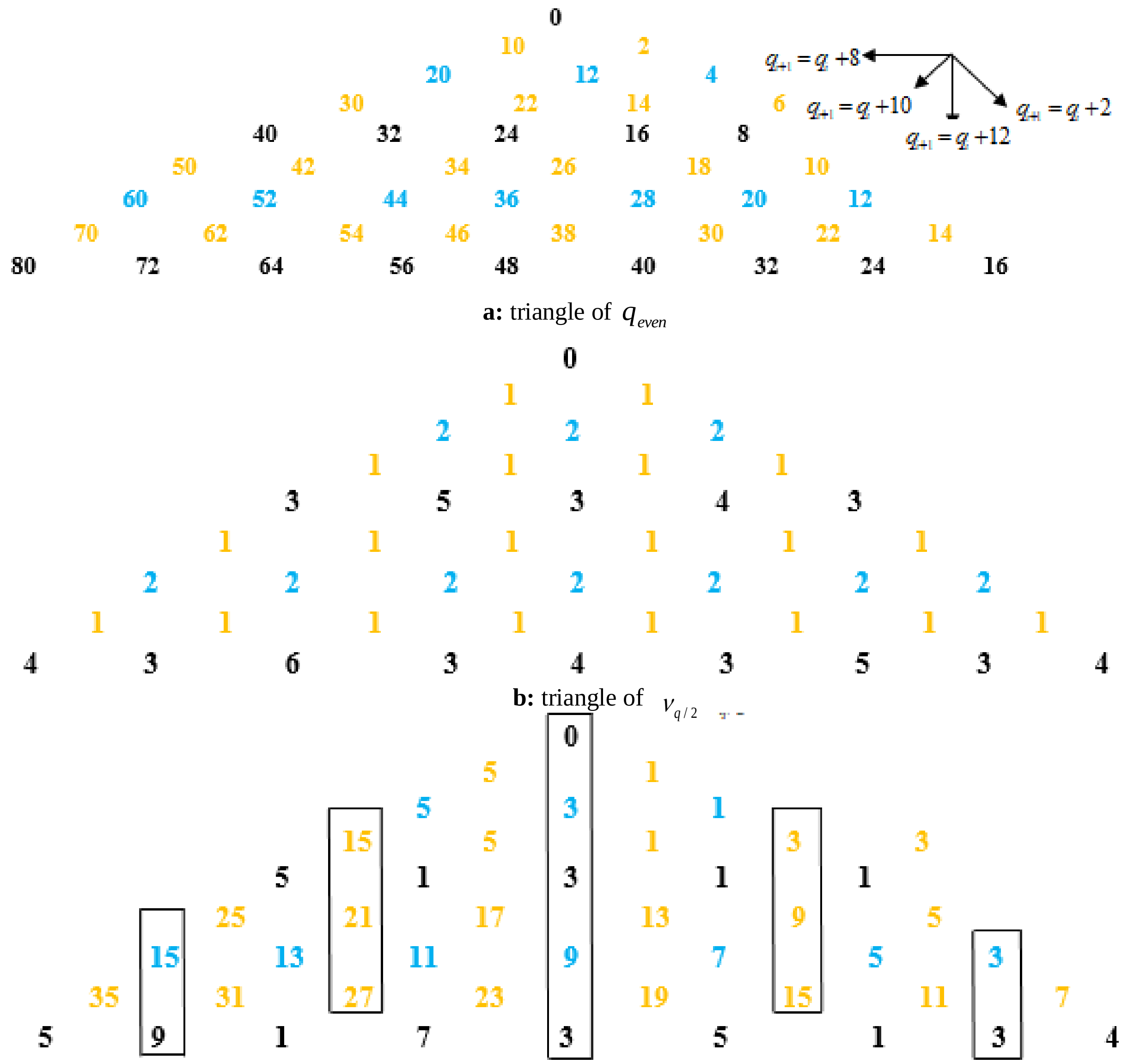
the formation of so-called branching nodes of new sequences $\{\theta_{new} \cdot 2^n\}_{n=0}^{\infty}$ with the parameter θ_{new} is carried out if the numbers θ_{new} are not divisible by κ. Such nodes can be considered active. Nodes with multiples of a in the numbers $\theta_{new} = \theta_a$ are inactive since the Jacobsthal numbers $J^{\pm}_{\theta_a,n}$ of the points of sequences $\{\theta_{new} \cdot 2^n\}_{n=0}^{\infty}$ are fractional [7]. The members of the sequences $\{\theta_{new} \cdot 2^n\}_{n=0}^{\infty}$ are formed by doubling the values.

Consider the regularities in the Collatz problem. In the $n \to 0$ direction, branching nodes play the role of merging nodes if equalities are satisfied at the corresponding points of the sequences $\{\theta \cdot 2^n\}_{n=0}^{\infty}$.

$$a \cdot J^{\pm}_{\kappa,\theta,n} + 1 = \theta \cdot 2^n \text{ (a) and } a \cdot J^{\pm}_{\kappa,\theta,n} - 1 = \theta \cdot 2^n \text{ (b).} \quad (3.6)$$

In (3.6), according to rule (3.6a), the sequence $C^{+}_{\kappa,q}$ is formed, and according to rule (3.6b), the sequence $C^{+}_{\kappa,q}$ is formed.

In both cases (3.6a) and (3.6b), even numbers are transformed by the same rule (3.4). Therefore, if even numbers are grouped in the form of a triangle (structuring even numbers in the form of a square matrix, as discussed in [7]) in Fig.5a, the regularities of grouping values $\nu_{q/2}$ are reflected in the form of rows of the triangle in Fig.5b.



a: triangle of q_{even}

b: triangle of $\nu_{q/2}$

c: triangle of q_{odd}

Fig. 5 The rows of the triangle

Here, the yellow colour highlights rows with the value $\nu_{q/2} = 1$, the blue colour highlights rows with the value $\nu_{q/2} = 2$, and the black colour highlights rows with the values $\nu_{q/2} \geq 3$. The inset displays the regularities of changes in the values of even numbers in the directions that form the triangle. Rows of the triangle in Fig. 5c are formed from odd numbers q_{odd}, which conclude the transformations (3.4).

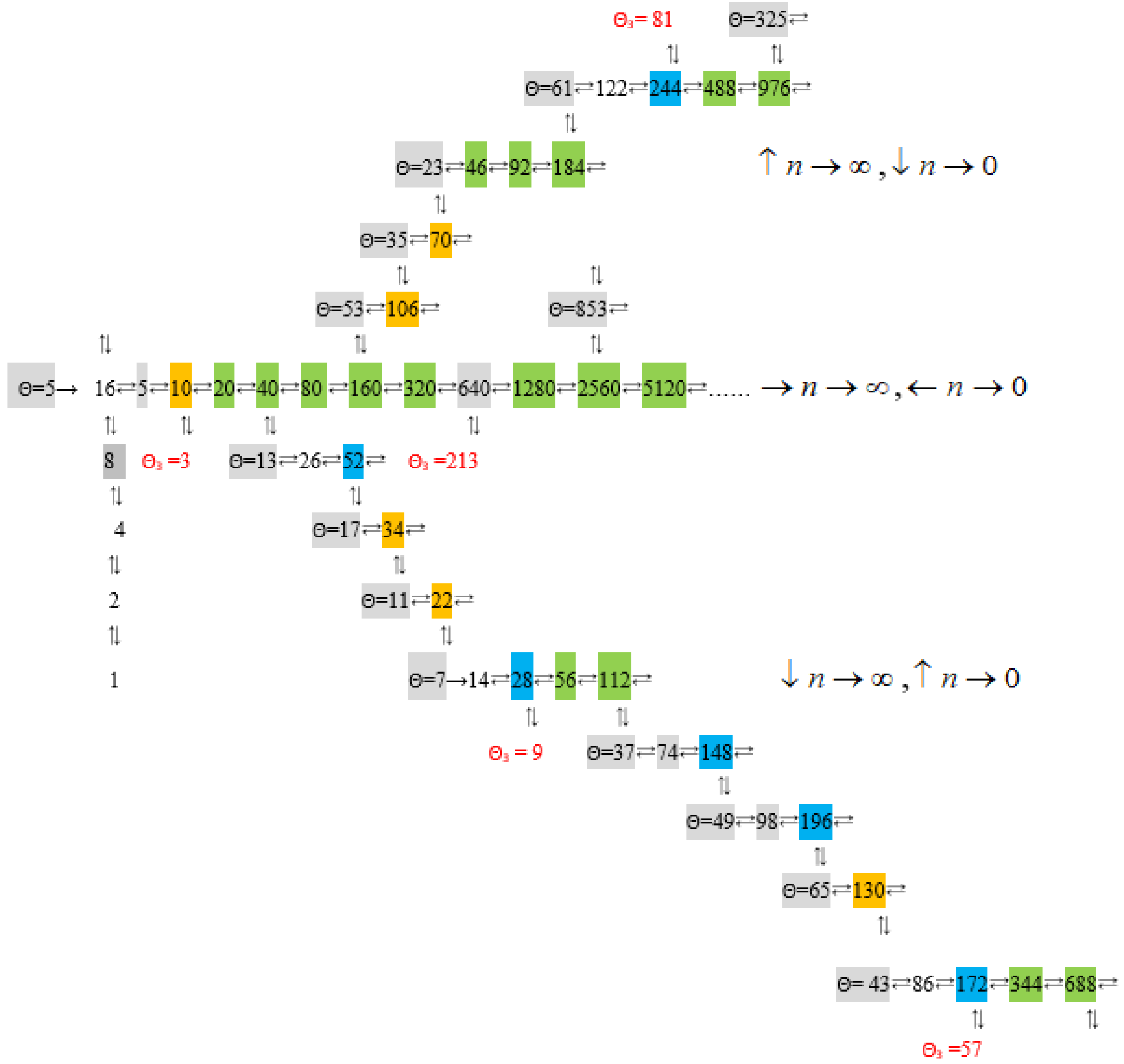


Fig. 6 Transformation of even numbers in Jacobsthal and Collatz

Illustration of the regularities in the formation and transformation of even numbers in Jacobsthal ($n \to +\infty$) and Collatz ($n \to 0$) sequences is presented in Fig. 6 for the sequence $\{5 \cdot 2^n\}_{n=0}^{\infty}$. In the case of $a = 3$, nodes of the $\{\theta \cdot 2^n\}_{n=0}^{\infty}$ sequences are periodically formed with powers having a period of $T_n = 2^2$, thus the Jacobsthal sequence takes the form:

$$J^{+}_{3,5,n}: \quad 3,\ 13,\ 53,\ 213,\ 853,\ 3413,\ 13653,\ \ldots \to \quad J^{+}_{3,5,n+2} = 4J^{+}_{3,5,n} + 1 \tag{3.7}$$

where red highlights indicate multiples of three, and neighbouring numbers are calculated as follows:

$$\left(\left(\left\{\begin{matrix}3\\9\\57\\\ldots\end{matrix}\right\}4+1\right)4+1\right)4+1=\left\{\begin{matrix}(((3\cdot4+1)4+1)4+1)=213\\(((9\cdot4+1)4+1)4+1)=597\\(((57\cdot4+1)4+1)4+1)=3669\\\ldots\end{matrix}\right\},\left(\left(\left\{\begin{matrix}213\\597\\3669\\\ldots\end{matrix}\right\}4+1\right)4+1\right)4+1=\left\{\begin{matrix}(((213\cdot4+1)4+1)4+1)=13653\\(((597\cdot4+1)4+1)4+1)=38229\\(((3669\cdot4+1)4+1)4+1)=234837\\\ldots\end{matrix}\right\}\ldots(3.8)$$

Even numbers of the Collatz sequence in Fig. 6, for which condition (3.5) equals neighbouring odd numbers in columns (3.8), form a periodic structure with branches of the type:

$$\begin{matrix}\cdots\rightleftarrows172\rightleftarrows86\rightleftarrows43\rightleftarrows130\rightleftarrows65\rightleftarrows196\rightleftarrows98\rightleftarrows49\rightleftarrows148\rightleftarrows74\rightleftarrows37\rightleftarrows112\rightleftarrows56\rightleftarrows28\rightleftarrows\cdots\\ \upharpoonleft\downharpoonright \qquad\qquad\qquad \upharpoonleft\downharpoonright\\ 57\rightleftarrows114\rightleftarrows228\rightleftarrows456\rightleftarrows\cdots\rightleftarrows57\cdot\theta^{\tau}\rightleftarrows\cdots \qquad 9\rightleftarrows18\rightleftarrows36\rightleftarrows72\rightleftarrows\cdots\rightleftarrows9\cdot\theta^{\tau}\rightleftarrows\cdots\end{matrix} \tag{3.9}$$

$$\cdots\rightleftarrows 406\rightleftarrows 203\rightleftarrows 610\rightleftarrows 305\rightleftarrows 916\rightleftarrows 458\rightleftarrows 299\rightleftarrows 688\rightleftarrows 244\rightleftarrows 172\rightleftarrows\cdots$$

$$\upharpoonleft\downharpoonright \qquad\qquad\qquad \upharpoonleft\downharpoonright$$

$$135\rightleftarrows 270\rightleftarrows 540\rightleftarrows 1080\rightleftarrows\cdots\rightleftarrows 135\cdot\theta^{\tau}\rightleftarrows\cdots \qquad 57\rightleftarrows 114\rightleftarrows 228\rightleftarrows 456\rightleftarrows\cdots\rightleftarrows 57\cdot\theta^{\tau}\rightleftarrows\cdots \qquad (3.10)$$

$$\cdots\rightleftarrows 3844\rightleftarrows 1922\rightleftarrows 911\rightleftarrows 2884\rightleftarrows 1442\rightleftarrows 721\rightleftarrows 2164\rightleftarrows 1082\rightleftarrows 541\rightleftarrows 1624\rightleftarrows 812\rightleftarrows 406\rightleftarrows\cdots$$

$$\upharpoonleft\downharpoonright \qquad\qquad\qquad \upharpoonleft\downharpoonright$$

$$1281\rightleftarrows 2562\rightleftarrows 5124\rightleftarrows\cdots\rightleftarrows 1281\cdot\theta^{\tau}\rightleftarrows\cdots \qquad 135\rightleftarrows 270\rightleftarrows 540\rightleftarrows\cdots\rightleftarrows 135\cdot\theta^{\tau}\rightleftarrows\cdots \qquad (3.11)$$

superposition of which, in the $n\to\infty$ direction, forms the so-called Jacobsthal tree [7], known in the literature [22] as the Collatz tree. However, sequences (3.9) - (3.11) do not indicate the existence of clear patterns of transformations $q_{even}/2$.

Let's analyse the statement formulated in paragraph 10 [21]: «*In comparison with the function* $5n+1$ *that has no elements on the trunk of the Collatz tree and…*». For this, we will provide a fragment of the tree of this transformation:

$$1\cdot2^0\rightleftarrows1\cdot2^1\rightleftarrows1\cdot2^2\rightleftarrows1\cdot2^3\rightleftarrows1\cdot2^4\rightleftarrows1\cdot2^5\rightleftarrows1\cdot2^6\rightleftarrows1\cdot2^7\rightleftarrows1\cdot2^8\rightleftarrows1\cdot2^9\rightleftarrows1\cdot2^{10}\rightleftarrows1\cdot2^{11}\rightleftarrows1\cdot2^{12}\rightleftarrows1\cdot2^{13}\rightleftarrows \qquad (3.12)$$

$$\upharpoonleft\downharpoonright \qquad \upharpoonleft\downharpoonright \qquad \upharpoonleft\downharpoonright \qquad \upharpoonleft\downharpoonright$$

$$0 \qquad 3=16\cdot0+3 \qquad 51=16\cdot3+3 \qquad 819=16\cdot51+3$$

We see that on the root sequence $\{1\cdot2^n\}_{n=0}^{\infty}$, there are points with Jacobsthal numbers $0, 3, 51, 819, \ldots$, which for the task $5n+1$ form active branching nodes with similar transformation patterns (3.9)-(3.11) for even numbers in them.

$$\rightleftarrows 1511176\rightleftarrows 755588\rightleftarrows 377794\rightleftarrows 188897\rightleftarrows 944486\rightleftarrows 472243\rightleftarrows 2361216\rightleftarrows 1180608\rightleftarrows 59030 \qquad (3.13)$$

$$\upharpoonleft\downharpoonright \qquad\qquad\qquad \upharpoonleft\downharpoonright$$

$$302245 \qquad\qquad\qquad 118060$$

Conclusions

It is shown that the sequences proposed by David Bařina are not equivalent to classical Collatz problem, but represent a separate series. Reasoned functions with sign switching (2.7) made it possible not only to achieve unambiguity in the construction of Collatz fractals, but also to implement for the first time in MathCAD codes the algorithm for calculating the Collatz sequences themselves (Fig. 2.4). Here are the graphs of the sequences themselves, as well as the functions that were used for this together with the algorithm. We see that, unlike other algorithms, the one presented in Fig.2.4 is extremely simple. The probabilistic model of the Collatz problem proposed by the author of the paper [21] is not confirmed.

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РЕКУРЕНТНІ ЧИСЛА ЯКОБСТАЛЯ ЯК ПЛАТФОРМА ПЕРЕТВОРЕНЬ *К*·Q±1

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Анотація. У статті досліджується роль рекурентних чисел Якобсталя у формуванні статистичних закономірностей моделі гіпотези натуральних чисел q ∈ N у загальній задачі к·q±1, де к=1,3,5,…. Вперше запропоновано модель структурування множини натуральних чисел у вигляді множини послідовностей виду $\theta\cdot 2^n$, де θ приймає непарні значення 1,3,5,…, а n – натуральні числа, починаючи з нуля. Розроблено діаграму розгалуження та злиття таких послідовностей у напрямку загального часу зупинки tst, де tst→∞. На основі цієї моделі вперше встановлено закономірності формування послідовностей чисел, що мають однакову довжину у послідовності Коллатца CSq. Отримані результати можуть бути використані для подальшого аналізу арифметичних перетворень та властивостей натуральних чисел у контексті теорії чисел.

Ключові слова: рекурентне число, гістограма, число Якобсталя, загальний час зупинки, гіпотеза Коллатца, функція розподілу ймовірностей.