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Solution of the Dynamic Problem of Thermoelasticity in Stresses for a Rectangular Beam

Musii R. S.

Lviv Polytechnic National University,
12 S. Bandera str., 79013, Lviv, Ukraine

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A methodology for constructing a solution to the two-dimensional dynamic problem of thermoelasticity in stresses for a rectangular beam is proposed. The system of stress equations describing the plane-deformed state of the beam under nonstationary thermal and mechanical loadings is chosen as the initial one. The methodology is based on the approximation of the distributions of all components of the dynamic stress tensor by cubic polynomials along the thickness coordinate of the beam. As a result, the system of initial two-dimensional nonstationary equations for these components is reduced to a system of one-dimensional nonstationary equations for the thickness-integral characteristics (analogous to forces and moments) of these components. The re-approximation of their distributions along the transverse coordinate of the beam by cubic polynomials is used. As a result, a system of ordinary differential equations with respect to the time variable was obtained for the integral characteristics of the analogs of forces and moments. Taking into account the initial conditions for these integral characteristics, the general solutions of the Cauchy problems for the integral characteristics of forces and moments are found using the Laplace transform in the time variable. The expressions of the components of the dynamic stress tensor and the stress intensity are derived from the expressions obtained in this way. A criterion for evaluating the bearing capacity of a beam is proposed. The resonant frequencies of four types for a beam are identified, which make it possible to study the resonant effects in a beam under nonstationary thermal and mechanical loadings.

Keywords: *beam; nonstationary thermal and mechanical loadings; plane-deformed state; stress tensor components; stress intensity; resonant frequencies.*

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1. Introduction

Plate-like structural elements are often subjected to intense nonstationary thermal and mechanical loadings during operation. To assess their bearing capacity, it is necessary to solve dynamic problems of thermoelasticity in stresses. This makes it possible to determine the stress intensity σ_i. According to the Huber–Mises criterion: $\sigma_i \leqslant \sigma_d$ [1], where σ_d is the dynamic limit of elastic deformation of the element material. The formulation of the problem of thermoelasticity in stresses was considered, in particular, in [2–5], where solutions are primarily presented for static problems and, in some cases, for certain one-dimensional dynamic problems. For a more complete study of the thermoelastic state of structural elements, it is necessary to address two-dimensional problems. In studies [6,7], the governing systems of equations for two-dimensional dynamic problems of plates and cylindrical bodies have been derived. The aim of this work is to construct a solution to the two-dimensional dynamic problem of thermoelasticity for a beam in terms of stresses under nonstationary and thermal loadings.

2. The relations of the dynamic thermoelasticity problem for a beam

In the Cartesian coordinate system $Ox_1x_2x_3$, we consider a rectangular beam of thickness $2h$ and width $2d$. The material of the beam is homogeneous and isotropic. The beam is subjected to unsteady volumetric force $\boldsymbol{F} = \{F_1; 0; F_3\}$ and the temperature field T. Their expressions do not depend on the coordinate x_2, and the surfaces of the beam $x_3 = \pm h$ and $x_1 = \pm d$ are free from surface force load. Under these conditions, we have a plane-deformed state of the beam. As a starting point, we choose the system of equations of the plane dynamic problem of thermoelasticity in stresses [7, 8]

$$\Delta_1\Psi - \frac{1}{C_1^2}\frac{\partial^2\Psi}{\partial t^2} = 2\rho\alpha\frac{1+\nu}{1-\nu}\frac{\partial^2 T}{\partial t^2} - \frac{\alpha E}{1-\nu}\Delta_1 T - \frac{1}{1-\nu}\left(\frac{\partial F_3}{\partial x_3} + \frac{\partial F_1}{\partial x_1}\right), \tag{1}$$

$$\Delta_1\sigma_{13} - \frac{1}{C_2^2}\frac{\partial^2\sigma_{13}}{\partial t^2} = -\frac{\partial^2\Psi}{\partial x^1\partial x_3} - \frac{\partial F_1}{\partial x_1} - \frac{\partial F_3}{\partial x_3}, \tag{2}$$

$$\frac{\partial^2\sigma_{11}}{\partial x_1^2} - \frac{1}{C_3^2}\frac{\partial^2\sigma_{11}}{\partial t^2} = -\frac{\partial^2\sigma_{13}}{\partial x_1\partial x_3} - \frac{\partial F_1}{\partial x_1} + \frac{\partial^2}{\partial t^2}\left[\rho\alpha(1+\nu)T - \rho\frac{\nu(1+\nu)}{E}\Psi\right], \tag{3}$$

$$\sigma_{33} = \Psi - \sigma_{11}, \quad \sigma_{22} = \nu\Psi - \alpha E T. \tag{4}$$

Here, $\Psi = \sigma_{11} + \sigma_{33}$, $C_1^2 = (1-\nu)(1+\nu)^{-1}(1-2\nu)^{-1}\rho^{-1}E$, $C_2^2 = E/\left[2\left(\rho(1-\nu)\right)\right]$, C_1 and C_2 are propagation rates of elastic waves of expansion and deformation, $\Delta_1 = \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_3^2}$, $C_3^2 = 2C_2^2$, α, ν are the coefficients of linear thermal expansion and Poisson's ratio, E is the Young's modulus, ρ is the density, t is time. The initial conditions at $t = 0$ on the defining functions are equal to

$$\sigma_{11} = \sigma_{13} = \Psi = \sigma_{22} = \sigma_{33} = 0, \quad \dot\sigma_{11} = -\frac{\alpha E\dot T}{(1-2\nu)},$$

$$\dot\sigma_{13} = 0, \quad \dot\sigma_{22} = -\frac{\alpha E\dot T}{(1-2\nu)}, \quad \dot\sigma_{33} = -\frac{\alpha E\dot T}{(1-2\nu)}, \quad \dot\Psi = -\frac{2\alpha E\dot T}{(1-2\nu)}. \tag{5}$$

Here, $\dot T$, $\dot\Psi$, $\dot\sigma_{11}$, $\dot\sigma_{13}$, $\dot\sigma_{22}$, $\dot\sigma_{33}$ are the time derivatives t of the determining functions.

When the faces of the beam are free of external forces, the boundary conditions on the surfaces $x_3 = \pm h$ take the following form:

$$\frac{\partial^2\Psi^\pm}{\partial x_1^2} - \frac{1}{C_4^2}\frac{\partial^2\Psi^\pm}{\partial t^2} = \rho\alpha(1+\nu)\frac{\partial^2\Psi^\pm}{\partial t^2} - \frac{\partial F^\pm}{\partial x_1} - \frac{\partial}{\partial x_1}\left(\frac{\partial\sigma_{13}}{\partial x_3}\right)^\pm, \quad \sigma_{13}^\pm = 0, \quad \sigma_{11}^\pm = \Psi^\pm. \tag{6}$$

Accordingly, on the surfaces $x_1 = \pm d$ it is written as:

$$\frac{\partial^2\Psi_*^\pm}{\partial x_3^2} - \frac{1}{C_4^2}\frac{\partial^2\Psi_*^\pm}{\partial t^2} = \rho\alpha(1+\nu)\frac{\partial^2 T_*^\pm}{\partial t^2} - \frac{\partial F_{*3}^\pm}{\partial x_3} - \frac{\partial}{\partial x_3}\left(\frac{\partial\sigma_{*13}}{\partial x_1}\right)^\pm, \quad \sigma_{*13}^\pm = 0; \sigma_{*33}^\pm = \Psi_*^\pm. \tag{7}$$

Here,

$$C_4^2 = 2(1-\nu)^{-1}C_2^2, \quad T^\pm = T(x_1, \pm h, t), \quad F_i^\pm = F_i(x_1, \pm h, t),$$

$$\overset{\pm}{\sigma}_{*ik} = \sigma_{ik}\left(\pm d; x_3; t\right), \quad T_*^\pm = T\left(\pm T; x_3; t\right), \quad F_{*i}^\pm = F_i\left(\pm T; x_3; t\right).$$

The symbol "$*$" in the lower indices of the quantities means their values on the surfaces $x_1 = \pm d$.

The solution of the original problem (1)–(4) is reduced to the joint solution of the two interconnected wave equations (1) and (2) and the following solvable equations (3) at the initial (5) and boundary (6)–(7) conditions.

3. Methodology for constructing the solution

To solve the system of equations (1)–(4), the functions ψ and σ_{13} are approximated by the thickness coordinate x_3 with cubic polynomials

$$\psi = \sum_{j=1}^{4}\psi_{j-1}(x_1,t)\,x_3^{j-1}, \quad \sigma_{13} = \sum_{j=1}^{4}\alpha_{13(j-1)}(x_1,t)\,x_3^{j-1}. \tag{8}$$

The coefficients ψ_{j-1} and $\alpha_{13(j-1)}$ of the approximation polynomials (8) are represented by the thickness coordinate x_3 integral characteristics (analogs of forces and moments) of the functions ψ and σ_{13}

$$N = \frac{1}{2h}\int_{-h}^{h}\psi\,dx_3, \quad M = \frac{3}{2h^2}\int_{-h}^{h}\psi x_3 dx_3, \quad N_{13} = \frac{1}{2h}\int_{-h}^{h}\sigma_{13}dx_3, \quad M_{13} = \frac{3}{2h^2}\int_{-h}^{h}\sigma_{13}x_3 dx_3 \tag{9}$$

and the boundary values of these functions on the surfaces $x_3 = \pm h$ are given. We obtain the expressions

$$\psi_{(0)} = \frac{3}{2}N - \frac{1}{4}\psi_*,\ \psi_{(1)} = \frac{5}{2h}M - \frac{3}{4h}\psi_{**},\ \psi_{(2)} = \frac{3}{4h^2}\psi_* - \frac{3}{2h^2}N,\ \psi_{(3)} = \frac{5}{4h^3}\psi_{**} - \frac{5M}{2h^2}\alpha_{13(0)} = \frac{3}{2}N_{13},$$

$$\alpha_{13(1)} = \frac{5}{2h}M_{13};\quad \alpha_{13(2)} = -\frac{3}{2h^2}N_{13};\quad \alpha_{13(3)} = -\frac{5}{2h^3}M_{13}. \tag{10}$$

Here, $\psi_* = \psi^+ + \psi^-$, $\psi_{**} = \psi^+ - \psi^-$. The system of equations for determining the integral characteristics of N, M, N_{13}, M_{13} is obtained by integrating equations (1), (2) according to formulas (9) using expressions (8) is as follows:

$$\left(\frac{\partial^2}{\partial x_1^2} - \frac{1}{c_1^2}\frac{\partial^2}{\partial t^2} - \frac{3}{h^2}\right)N = \Phi_1 - \frac{3}{2h^2}\left(\psi^+ + \psi^-\right),\ \left(\frac{\partial^2}{\partial x_1^2} - \frac{1}{c_1^2}\frac{\partial^2}{\partial t^2} - \frac{15}{h^2}\right)M = \Phi_2 - \frac{15}{2h^2}\left(\psi^+ - \psi^-\right),$$

$$\left(\frac{\partial^2}{\partial x_1^2} - \frac{1}{c_2^2}\frac{\partial^2}{\partial t^2} - \frac{3}{h^2}\right)N_{13} = \frac{1}{2h}\left[\frac{\partial}{\partial x_1}\left(\psi^+ - \psi^-\right) + \left(F_1^+ - F_1^-\right)\right] + \frac{\partial F_3^{(1)}}{\partial x_1},$$

$$\left(\frac{\partial^2}{\partial x_1^2} - \frac{1}{c_2^2}\frac{\partial^2}{\partial t^2} - \frac{15}{h^2}\right)M_{13} = \frac{3}{2h}\left[\frac{\partial}{\partial x_1}\left(\psi^+ + \psi^-\right) + \left(F^+ + F^-\right) - \frac{3}{h}\left(\frac{\partial N}{\partial x_1} + F_1^{(1)}\right)\right] + \frac{\partial F_3^{(2)}}{\partial x_1}. \tag{11}$$

The initial conditions will be as follows

$$N(x_1,0) = 0,\quad M(x_1,0) = 0,\quad N_{13}(x_1,0) = 0,\quad M_{13}(x_1,0) = 0,\quad \dot{N}(x_1,0) = -\frac{2\alpha E}{1-2\nu}\dot{T}_1(x_1,0),$$

$$\dot{M}(x_1,0) = -\frac{2\alpha E}{1-2\nu}\dot{T}_2(x_1,0),\quad \dot{N}_{13}(x_1,0) = 0,\quad \dot{M}_{13}(x_1,0) = 0. \tag{12}$$

The functions Φ_1 and Φ_2 in the equations (11) are as follows

$$\Phi_1 = 2\rho\alpha\frac{1+\nu}{1-\nu}\frac{\partial^2 T_1}{\partial t^2} - \frac{\alpha E}{1-\nu}\left\{\frac{\partial^2 T_1}{\partial x_1^2} + \frac{1}{2h}\left[\left(\frac{\partial T}{\partial x_3}\right)^+ - \left(\frac{\partial T}{\partial x_3}\right)^-\right]\right\} - \frac{1}{1-\nu}\left[\frac{1}{2h}\left(F_3^+ - F_3^-\right) + \frac{\partial F_1^{(1)}}{\partial x_1}\right],$$

$$\Phi_2 = 2\rho\alpha\frac{1+\nu}{1-\nu}\frac{\partial^2 T_2}{\partial t^2} - \frac{\alpha E}{1-\nu}\left\{\frac{\partial^2 T_2}{\partial x_1^2} + \frac{3}{2h}\left[\left(\frac{\partial T}{\partial x_3}\right)^+ - \left(\frac{\partial T}{\partial x_3}\right)^-\right] - \frac{3}{2h^2}\left(T^+ - T^-\right)\right\}$$
$$- \frac{1}{1-\nu}\left[\frac{3}{2h}(F_3^+ + F_3^-) - F_3^{(1)} + \frac{\partial F_1^{(2)}}{\partial x_1}\right].$$

The functions Φ_1 and Φ_2 in the equations (11) are as follows For the boundary values of $\psi^\pm$ of the function ψ on the surfaces of $x_1 = \pm d$ of the bar, we obtain the equation

$$\frac{\partial^2\psi^\pm}{\partial x_1^2} - \frac{1}{c_4^2}\frac{\partial^2\psi^\pm}{\partial t^2} = \rho\alpha(1+\nu)\frac{\partial^2 T^\pm}{\partial t^2} - \frac{\partial F_1^\pm}{\partial x_1} + \frac{1}{h}\frac{\partial}{\partial x_1}(5M_{13} \pm 3N_{13}^{(d)}) \tag{13}$$

and initial conditions as

$$\psi^\pm(x_1,0) = \psi(x_1,\pm h,0) = 0,\quad \dot{\psi}^\pm(x_1,0) = \dot{\psi}(x_1,\pm h,0) = -\frac{2\alpha E}{1-2\nu}\dot{T}^\pm(x_1,\pm h,0). \tag{14}$$

Taking into account the conditions for conjugation of the values of the functions ψ and σ_{13} at the vertices of the rectangle of the cross-section of the beam [9] on the functions N, M, N_{13}, M_{13}, $\Psi^\pm$ along the coordinate x_1, we obtain the following boundary conditions

$$\frac{d^2N(\pm d,t)}{dt^2} + \frac{3c_4^2}{h^2}N(\pm d,t) = \frac{c_4^2}{2h}\left[F_3^+(\pm d,t) - F_3^-(\pm d,t)\right] - c_4^2\rho\alpha(1+\nu)\frac{d^2T_1(\pm d,t)}{dt^2}, \tag{15}$$

$$\frac{d^2M(\pm d,t)}{dt^2} + \frac{15c_4^2}{h^2}M(\pm d,t) = 3c_4^2\left\{\frac{1}{2}\left[F_3^+(\pm d,t) - F_3^-(\pm d,t)\right] + F_3^{(1)}(\pm d,t)\right\}$$
$$+ \frac{3c_4^2}{2h}\left(\frac{dN_{13}}{dx_1}\right)_{x_1=\pm d} - c_4^2\rho\alpha(1+\nu)\frac{d^2T_2(\pm d,t)}{dt^2}, \tag{16}$$

$$\psi^\pm(\pm d,t) = 0,\quad N_{13}(\pm d,t) = 0,\quad M_{13}(\pm d,t) = 0.$$

In accordance with the inhomogeneous boundary conditions (15)–(16) on the functions N and M, we present the solutions of the first two equations of the system (11) in the form

$$N(x_1,t) = \frac{1}{2}\left\{\left[N_*(d,t) + N_{**}(-d,t)\right] + \frac{x_1}{d}\left[N_*(d,t) - N_{**}(-d,t)\right]\right\} + N_0(x_1,t), \tag{17}$$

$$M(x_1,t) = \frac{1}{2}\left\{\left[M_*(d,t) + M_{**}(-d,t)\right] + \frac{x_1}{d}\left[M_*(d,t) - M_{**}(-d,t)\right]\right\} + M_0(x_1,t). \tag{18}$$

Then the functions $N_0(x_1,t)$ and $M_0(x_1,t)$ are defined from the following equations

$$\left(\frac{d^2}{dx_1^2} - \frac{1}{c_1^2}\frac{d^2}{dt^2} - \frac{3}{h^2}\right) N_0 = \Phi_1 - \frac{3}{2h^2}(\psi^+ + \psi^-) + \left(\frac{1}{c_1^2}\frac{d^2}{dt^2} + \frac{3}{h^2}\right)\frac{1}{2}\left[(N_* + N_{**}) + \frac{x_1}{d}(N_* - N_{**})\right],$$

$$\left(\frac{d^2}{dx_1^2} - \frac{1}{c_1^2}\frac{d^2}{dt^2} - \frac{15}{h^2}\right) M_0 = \Phi_2 - \frac{15}{2h^2}(\psi^+ - \psi^-) + \left(\frac{1}{c_1^2}\frac{d^2}{dt^2} + \frac{15}{h^2}\right)\frac{1}{2}\left[(M_* + M_{**}) + \frac{x_1}{d}(M_* + M_{**})\right] \tag{19}$$

at homogeneous boundary conditions on the surfaces $x_1 = \pm d$. Here, $N_* = N(d,t)$, $N_{**} = N(-d,t)$, $M_* = M(d,t)$, $M_{**} = M(-d,t)$.

The system of interdependent equations (11) and equations (13) and (19) are solved using the approximation of the functions N, M, N_{13}, M_{13}, $\psi^\pm$, σ_{11} in the coordinate x_1 by cubic polynomials, i.e. we have the expressions

$$N = \sum_{j=1}^{4} b_N^{(j-1)}(t)\, x_1^{j-1}, \qquad M = \sum_{j=1}^{4} b_M^{(j-1)}(t)\, x_1^{j-1}, \tag{20}$$

$$N_{13} = \sum_{j=1}^{4} b_{13N}^{(j-1)}(t)\, x_1^{j-1}, \qquad M_{13} = \sum_{j=1}^{4} b_{13M}^{(j-1)}(t)\, x_1^{j-1}, \tag{21}$$

$$\psi^\pm = \sum_{j=1}^{4} c_{(j-1)}^{\pm}(t)\, x_1^{j-1}, \qquad \sigma_{11} = \sum_{j=1}^{4} b_{11}^{(j-1)}(t)\, x_1^{j-1}. \tag{22}$$

The coefficients $b_N^{(j-1)}$, $b_M^{(j-1)}$, $b_{13N}^{(j-1)}$, $b_{13M}^{(j-1)}$, $c_{j-1}^{\pm}$, $b_{11}^{(j-1)}$ of the approximation polynomials (20)–(21) are represented by the characteristics integral along the coordinate x_1

$$R^{(n)}(t) = \frac{2n-1}{2d^{n-1}}\int_{-d}^{d} R(x_1,t)\, x_1^{n-1} dx_1, \qquad (n = 1,2) \tag{23}$$

of functions $R(x_1,t) = \{N, M, N_{13}, M_{13}, \psi^\pm, \sigma_{11}\}$ and the corresponding boundary conditions on the surfaces $x_1 = \pm d$ of the beam. For the coefficients $b_N^{(j-1)}$, $b_M^{(j-1)}$, $b_{13N}^{(j-1)}$, $b_{13M}^{(j-1)}$, $c_{j-1}^{\pm}$, $b_{11}^{(j-1)}$ $(j = \bar{1},\bar{4})$ of polynomials (20)–(22) we get the expressions

$$\begin{gathered}
b_N^{(0)} = \frac{3}{2}N^{(1)} - \frac{1}{4}(N_* + N_{**}), \quad b_N^{(1)} = \frac{5}{2d}N^{(2)} - \frac{3}{4d}(N_* - N_{**}),\\
b_N^{(2)} = \frac{3}{4d^2}(N_* + N_{**}) - \frac{3}{2d^2}N^{(1)}, \quad b_N^{(3)} = \frac{5}{4d^3}(N_* - N_{**}) - \frac{5}{2d^3}N^{(2)},\\
b_M^{(0)} = \frac{3}{2}M^{(1)} - \frac{1}{4}(M_* + M_{**}), \quad b_M^{(1)} = \frac{5}{2d}M^{(2)} - \frac{3}{4d}(M_* - M_{**}),\\
b_M^{(2)} = \frac{3}{4d^2}(M_* + M_{**}) - \frac{3}{2d^2}M^{(1)}, \quad b_M^{(3)} = \frac{5}{4d^3}(M_* - M_{**}) - \frac{5}{2d^2}M^{(2)},\\
b_{13N}^{(0)} = \frac{3}{2}N_{13}^{(1)}, \quad b_{13N}^{(1)} = \frac{5}{2d}N_{13}^{(2)}, \quad b_{13N}^{(2)} = -\frac{3}{2d^2}N_{13}^{(1)}, \quad b_{13N}^{(3)} = -\frac{5}{2d^3}N_{13}^{(2)},\\
b_{13M}^{(0)} = \frac{3}{2}M_{13}^{(1)}, \quad b_{13M}^{(1)} = \frac{5}{2d}M_{13}^{(2)}, \quad b_{13M}^{(2)} = -\frac{3}{2d^2}M_{13}^{(1)}, \quad b_{13M}^{(3)} = -\frac{5}{2d^3}M_{13}^{(2)},\\
c_0^{\pm} = \frac{3}{2}\psi^{\pm(1)}, \quad c_1^{\pm} = \frac{5}{2d}\psi^{\pm(2)}, \quad c_2^{\pm} = -\frac{3}{2d^2}\psi^{\pm(1)}, \quad c_3^{\pm} = -\frac{5}{2d^3}\psi^{\pm(2)},\\
b_{11}^{(0)} = \frac{3}{2}\sigma_{11}^{(1)}, \quad b_{11}^{(1)} = \frac{5}{2d}\sigma_{11}^{(2)}, \quad b_{11}^{(2)} = -\frac{3}{2d^2}\sigma_{11}^{(1)}, \quad b_{11}^{(3)} = -\frac{5}{2d^3}\sigma_{11}^{(2)}.
\end{gathered} \tag{24}$$

The systems of equations for determining the integral characteristics of (23) are obtained by averaging over the coordinate x_1 the equations (11), (13) and (19) and these equations multiplied by the coordinate x_1.

Then the system of equations for the integral characteristics of $N^1(t)$ and $N^2(t)$ of the function $N(x_1,t)$ are as follows

$$\begin{aligned}\left(\frac{\partial^2}{\partial t^2}+\frac{3c_1^2}{h^2}+\frac{3c_1^2}{d^2}\right)N^{(1)}&=\frac{3c_1^2}{2h^2}\left(\psi^{+(1)}+\psi^{-(1)}\right)+\frac{3}{2d^2}\left(N_*+N_{**}\right)-c_1^2\Phi_1^{(1)},\\ \left(\frac{\partial^2}{\partial t^2}+\frac{3c_1^2}{h^2}+\frac{15c_1^2}{d^2}\right)N^{(2)}&=\frac{3c_1^2}{2h^2}\left(\psi^{+(2)}+\psi^{-(2)}\right)+\frac{15}{2d^2}\left(N_*-N_{**}\right)-c_1^2\Phi_1^{(2)},\end{aligned}\tag{25}$$

and for the characteristics $M^1(t)$ and $M^2(t)$ of the function $M(x_1,t)$ is written as

$$\begin{aligned}\left(\frac{\partial^2}{\partial t^2}+\frac{15c_1^2}{h^2}+\frac{3c_1^2}{d^2}\right)M^{(1)}&=\frac{15c_1^2}{2h^2}\left(\psi^{+(1)}-\psi^{-(1)}\right)+\frac{3}{2d^2}(M_*+M_{**})-c_1^2\Phi_2^{(1)},\\ \left(\frac{\partial^2}{\partial t^2}+\frac{15c_1^2}{h^2}+\frac{15c_1^2}{d^2}\right)M^{(2)}&=\frac{15c_1^2}{2h^2}\left(\psi^{+(2)}-\psi^{-(2)}\right)+\frac{15}{2d^2}(M_*-M_{**})-c_1^2\Phi_2^{(2)}.\end{aligned}\tag{26}$$

Equations for finding the characteristics of $N_{13}^1(t)$ and $N_{13}^2(t)$ of the function $N_{13}(x_1,t)$ will be in form

$$\begin{aligned}\left(\frac{\partial^2}{\partial t^2}+\frac{3c_2^2}{h^2}+\frac{3c_2^2}{d^2}\right)N_{13}^{(1)}&=-\left[\frac{1}{2h}\left(F_1^{+(1)}-F_1^{-(1)}\right)+\frac{1}{2d}\left(F_{3*}^{(1)}-F_{3**}^{(1)}\right)\right]c_2^2,\\ \left(\frac{\partial^2}{\partial t^2}+\frac{3c_2^2}{h^2}+\frac{15c_2^2}{d^2}\right)N_{13}^{(2)}&=c_2^2\left[\frac{3}{2hd}\left(\psi^{+(1)}-\psi^{-(1)}\right)-\frac{1}{2h}\left(F_1^{+(2)}-F_1^{-(2)}\right)\right.\\ &\qquad\left.+\frac{3}{d}\left(F_3^{(1)}\right)^{(1)}-\frac{3}{2d}\left(F_{3*}^{(1)}+F_{3**}^{(1)}\right)\right].\end{aligned}\tag{27}$$

Accordingly, the system of equations for the characteristics $M_{13}^1(t)$ and $M_{13}^2(t)$ of the function $M_{13}(x_1,t)$ are written as

$$\begin{aligned}\left(\frac{\partial^2}{\partial t^2}+\frac{15c_2^2}{h^2}+\frac{3c_2^2}{d^2}\right)M_{13}^{(1)}&=-\frac{3c_2^2}{2h}\left(F_1^{+(1)}+F_1^{-(1)}\right)+\frac{3c_2^2}{2hd}\left(N_*-N_{**}\right)\\ &\qquad+\frac{3c_2^2}{h}\left(F_1^{(1)}\right)^{(1)}+c_2^2\left(F_{3*}^{(2)}-F_{3**}^{(2)}\right),\\ \left(\frac{\partial^2}{\partial t^2}+\frac{15c_2^2}{h^2}+\frac{15c_2^2}{d^2}\right)M_{13}^{(2)}&=\frac{9c_2^2}{2hd}(\psi^{+(1)}+\psi^{-(1)})-\frac{3c_2^2}{2h}\left(F_1^{+(2)}+F_1^{-(2)}\right)\\ &\qquad+\frac{9c_2^2}{2hd}\left(N_*+N_{**}\right)+\frac{9c_2^2}{hd}N^{(1)}+\frac{3c_2^2}{h}\left(F_1^{(1)}\right)^{(2)}\\ &\qquad-\frac{3c_2^2}{2d}\left(F_{3*}^{(2)}+F_{3**}^{(2)}\right)+\frac{3c_2^2}{d}\left(F_3^{(2)}\right)^{(1)}.\end{aligned}\tag{28}$$

For the integral characteristics $\psi^{\pm(n)}(t)$ of the boundary values $\psi^{\pm}(x_1,t)$ of the function ψ, the system of initial equations takes the form

$$\begin{aligned}\left(\frac{\partial^2}{\partial t^2}+\frac{3c_4^2}{d^2}\right)\psi^{\pm(1)}&=-c_4^2\rho\alpha(1+\nu)\frac{\partial^2T^{\pm(1)}}{\partial t^2}+\frac{c_4^2}{2d}\left(F_{1*}^{\pm}-F_{1**}^{\pm}\right),\\ \left(\frac{\partial^2}{\partial t^2}+\frac{15c_4^2}{d^2}\right)\psi^{\pm(2)}&=-c_4^2\rho\alpha(1+\nu)\frac{\partial^2T^{\pm(2)}}{\partial t^2}+\frac{3c_4^2}{2d}\left(F_{1*}^{\pm}+F_{1**}^{\pm}\right)\\ &\qquad-\frac{3c_4^2}{d}\left(F_1^{\pm}\right)^{(1)}+\frac{3c_4^2}{hd}\left(5M_{13}^{(1)}\pm3N_{13}^{(1)}\right).\end{aligned}\tag{29}$$

By averaging the equation (3) along the coordinate x_1, to find the integral characteristics of $\sigma_{11}^{(n)}(t)$ of the function σ_{11}, we obtain the following equations

$$\left(\frac{\partial^2}{\partial t^2}+\frac{3c_3^2}{d^2}\right)\sigma_{11}^{(1)}=\frac{c_3^2}{2d}\left(\frac{\partial\sigma_{13*}}{\partial x_3}-\frac{\partial\sigma_{13**}}{\partial x_3}\right)+\frac{c_3^2}{2d}(F_{1*}-F_{1**})-\frac{\partial^2}{\partial t^2}\left[\rho\alpha(1+\nu)c_3^2T^{(1)}-\rho c_3^2\frac{\nu(1+\nu)}{E}N\right],$$

$$\left(\frac{\partial^2}{\partial t^2}+\frac{15c_3^2}{d^2}\right)\sigma_{11}^{(2)}=\frac{3c_3^2}{2d}\left(\frac{\partial\sigma_{13*}}{\partial x_3}+\frac{\partial\sigma_{13**}}{\partial x_3}\right)+\frac{3c_3^2}{2d}\left(F_{1*}+F_{1**}\right)-\frac{3c_3^2}{d}F_1^{(1)}$$
$$+\frac{\partial^2}{\partial t^2}\left[\rho\alpha(1+\nu)c_3^2T^{(2)}-\rho c_3^2\frac{\nu(1+\nu)}{E}M\right]. \quad (30)$$

Here, the upper indices "+" and "−" refer to the surfaces $x_3=\pm1$, and the lower indices "$*$" and "$**$" refer to the surfaces $x_1=\pm d$ of the beam, respectively;

$$\Phi_s^{(n)}=\frac{2n-1}{2d^{n-1}}\int_{-d}^{d}\Phi_s x_1^{n-1}dx_1 \qquad (s,n=1,2),$$
$$F_k^{\pm(n)}=\frac{2n-1}{2d^{n-1}}\int_{-d}^{d}F_k^{\pm}x_1^{n-1}dx_1, \qquad \left(F_k^{(1)}\right)^{(n)}=\frac{2n-1}{2d^{n-1}}\int_{-d}^{d}F_k^{(1)}x_1^{n-1}dx_1.$$

Averaging according to the formula (23) the initial conditions (12) and (14) we obtain the following initial conditions for the functions $N^{(n)}$, $M^{(n)}$, $N_{13}^{(n)}$, $M_{13}^{(n)}$, $\psi^{\pm(n)}$, $\sigma_{11}^{(n)}$ $(n=1,2)$:

$$N^{(n)}(0)=0,\quad M^{(n)}(0)=0,\quad N_{13}^{(n)}(0)=0,\quad M_{13}^{(n)}(0)=0,$$
$$\psi^{\pm(n)}(0)=0,\quad \sigma_{11}^{(n)}(0)=0,\quad \dot N^{(n)}(0)=-\frac{2\alpha E}{1-2\nu}\dot T_1^{(n)}(0),$$
$$\dot M^{(n)}(0)=-\frac{2\alpha E}{1-2\nu}\dot T_2^{(n)}(0),\quad \dot N_{13}^{(s)}(0)=0,\quad \dot M_{13}^{(s)}(0)=0, \quad (31)$$
$$\psi^{\pm(s)}(0)=0,\quad \dot\psi^{\pm(s)}(0)=-\frac{2\alpha E}{1-2\nu}\dot T^{\pm(s)}(0),$$
$$\sigma_{11}^{(s)}(0)=0,\quad \dot\sigma_{11}^{(s)}(0)=-\frac{\alpha E}{1-2\nu}\dot T^{(s)}(0).$$

Applying the Laplace transform to the equations (25)–(30) taking into account the initial conditions (31), we find the transformants of the functions $R^{(n)}(t)$. Using the decomposition theorem and the convolution theorem, the originals of these functions are derived. Given the known functions $R^{(n)}(t)$, using the relations (20)–(22), we determine the functions N, M, N_{13}, M_{13}, $\psi^{\pm}$, σ_{11}, and by the formulas

$$\psi(x_1,x_3,t)=\frac{3}{2}\left(1-\frac{x_3^2}{h^2}\right)N(x_1,t)+\frac{5}{2}\left(\frac{x_3}{h}-\frac{x_3^3}{h^3}\right)M(x_1,t)-\frac{1}{4}\left(1-3\frac{x_3^2}{h^2}\right)\psi_*(x_1,t)$$
$$-\frac{1}{4}\left(3\frac{x_3}{h}-5\frac{x_3^3}{h^3}\right)\psi_{**}(x_1,t), \quad (32)$$

$$\sigma_{13}(x_1,x_3,t)=\frac{3}{2}\left(1-\frac{x_3^2}{h^2}\right)N_{13}(x_1,t)+\frac{5}{2}\left(\frac{x_3}{h}-\frac{x_3^3}{h^3}\right)M_{13}(x_1,t) \quad (33)$$

the expressions of the functions ψ and σ_{13} are derived. Here, $\Psi_*(x_1,t)=\Psi^+ + \Psi^-$, $\Psi_{**}(x_1,t)=\Psi^+ - \Psi^-$. Accordingly, the functions Ψ^{+-} and σ_{11} are found from the relations (22). Using the algebraic relations (4), the components σ_{22} and σ_{33} of the stress tensor are obtained.

1To predict the bearing capacity of a beam using the known components $\sigma_{11}(x_1,x_3,t)$, $\sigma_{13}(x_1,x_3,t)$, $\sigma_{22}(x_1,x_3,t)$, $\sigma_{33}(x_1,x_3,t)$ of the stress tensor and the temperature $T(x_1,x_3,t)$, we determine by the formula [2,4,6]

$$\sigma_i=\sqrt{(\sigma_{11}-\sigma_{22})^2+(\sigma_{22}-\sigma_{33})^2+(\sigma_{33}-\sigma_{11})^2+6\sigma_{13}^2}/\sqrt{2}$$

stress intensity σ_i and, according to the Huber–Mises criterion, compare it with the elastic deformation limit σ_d of the beam material.

4. Conclusion

The general solutions of the dynamic problem of thermoelasticity in stresses for a beam obtained on the basis of the above-proposed method make it possible to predict its bearing capacity under nonstationary thermal and mechanical loadings. This method allows us to evaluate the resonant effects of the thermoelastic behavior of a beam for four types of natural frequencies of its mechanical vibrations

$$\omega_1 = c_1\sqrt{\frac{3}{h^2}}, \quad \omega_2 = c_1\sqrt{\frac{15}{h^2}}, \quad \omega_3 = c_2\sqrt{\frac{3}{h^2}}, \quad \omega_4 = c_2\sqrt{\frac{15}{h^2}}$$

and is a scientific basis for choosing safe modes of temperature and force loads of structural elements in the form of a beam.

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Розв'язок динамічної задачі термопружності у напруженнях для бруса прямокутного перерізу

Мусій Р. С.

Національний університет "Львівська політехніка", вул. С. Бандери, 12, 79013, Львів, Україна

Запропоновано методику побудови розв'язку двовимірної динамічної задачі термопружності у напруженнях для бруса прямокутного перерізу. За вихідну вибрана система рівнянь у напруженнях, що описує плоскодеформований стан бруса за нестаціонарних температурних і силових дій. Методика ґрунтується на апроксимації розподілів всіх компонент тензора динамічних напружень кубічними поліномами за товщинною координатою бруса. У результаті система вихідних двовимірних нестаціонарних рівнянь на ці компоненти зведена до системи одновимірних нестаціонарних рівнянь на інтегральні за товщинною координатою характеристики (аналоги зусиль і моментів) даних компонент. Використано повторну апроксимацію їх розподілів за поперечною координатою бруса кубічними поліномами. У результаті записано систему звичайних диференціальних рівнянь за часовою змінною на інтегральні характеристики аналогів зусиль і моментів. Враховуючи початкові умови на ці інтегральні характеристики, знайдено з використанням перетворення Лапласа за часовою змінною загальні розв'язки задач Коші на інтегральні характеристики зусиль і моментів. За отриманими таким чином їх виразами записуються вирази компонент тензора динамічних напружень та інтенсивності напружень. Запропоновано критерій для оцінки несучої здатності бруса. Виявлено резонансні частоти чотирьох типів для бруса, які дають змогу дослідити резонансні ефекти у брусі за нестаціонарних температурних і силових дій.

Ключові слова: *брус; нестаціонарні температурні і силові дії; плоскодеформований стан; компоненти тензора напружень; інтенсивності напружень; резонансні частоти.*