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Path Integral Solutions for Extended Heston Models

Yanishevskyi V. S.

Lviv Polytechnic National University,
12 S. Bandera str., 79013, Lviv, Ukraine

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The path integral method developed in the author's previous work is applied to solve extended Heston models. A transition probability density is obtained for Heston models with stochastic interest rates described by Vasicek and CIR processes. Option pricing formulas are derived for both cases. The analytic solutions are valid when the Wiener process driving the interest rate is uncorrelated with the Wiener processes of the asset price and volatility. In the presence of such correlations, the model no longer admits an analytic solution, which is consistent with results reported in other studies.

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1. Introduction

Stochastic differential equation (SDE), also known as Langevin equations [1–6], are used to model different in their nature processes, like: financial, physical, and others. Stochastic elements of SDE are usually based on Wiener stochastic process (Brownian motion). Each SDE system has a corresponding Fokker–Planck equation (FP) for transition probability density of variable stochastic processes. Models based on SDE systems and respective Fokker–Planck equation are equivalent.

Solution to the FP equation is a quite effective using path integral approach [7–10]. In particular integral over Wiener measure is a path integral for the simplest stochastic equation which is given by Brownian motion [10, 11]. Path integral methods are well developed for one-dimensional systems for which path integrals are built based on solution of FP equation as well as using variable substitution in Wiener measure [9–11]. For stochastic processes of multiple variables with multiplicative Brownian motion there are a lot of inconsistencies between different approaches of building solution to FP equation using path integral method [12]. In work [12] a solution to multidimensional equation was built using path integral method. Based on this solution a solution to the two-dimensional Heston model of asset and derivatives pricing in financial engineering [13] was found. Given solution for transition probability density for [12] model coincides with the one found using Laplace and Fourier transforms for FP equation in [14].

The classical Heston model [13] is an extension to the Black-Scholes model in case of stochastic volatility, where volatility dynamics is given by Cox–Ingersoll–Ross (CIR) process. Heston model belongs to affine diffusion processes [13] for which the structure of characteristic functions is known. By substituting it into options price equation of the model, which is in fact an inverse Kolmogorov equation in relation to FP equation, we find the Heston solution.

We also consider Heston model extensions that take into account stochastic dynamics of interest rate. For interest rate we apply following stochastic processes [15–17] as Vasicek model, CIR process and others. Some of the Heston model extensions belong to the affine models, where solutions are found in a closed form by setting the form of characteristic function and substituting it into option equation [15–20]. It was noted that extended Heston models are more complex because of correlation with additional Wiener processes, that is why a closed-form solution does not always exist. In a number of works some approximate methods are proposed for solutions and also numeric modeling using Monte Carlo method is used [15, 20, 21]. A number of other extensions of Heston model are considered in the following works [22, 23].

In path integral method we consider Heston model extensions that contain stochastic process of interest rate. As opposed to approaches of mentioned works we will find solution to the FP equation for transition probability density of asset price and determine formula for option price. We will also show that in the case of correlation of Wiener process of interest rate with other stochastic processes, the path integrals do not have analytic solutions.

2. Construction of path integral for Fokker–Planck equation

Let us consider a stochastic model of multiple variables given by the system of SDE [1, 2, 4]:

$$dX_i(\tau) = A_i(X(\tau))\,d\tau + \sum_{j=1}^{n} B_{ij}(X(\tau))\,dW_j(\tau), \quad i \in \{1,\dots,n\}. \tag{1}$$

We use the following notation for the stochastic variables $X_i(\tau)$ and Wiener process increments $dW_i(\tau)$, $i \in \{1,\dots,n\}$, $\tau \in [t_0, t]$. Values $A_i(X(\tau))$, $i \in \{1,\dots,n\}$ denote components of drift vector that depend on the set of variables $X_i(\tau)$, $B_{ij}(X(\tau))$ — diffusion matrix of size $n \times n$, whose elements determine the local volatility of the stochastic variables ($(i,j) \in \{1,\dots,n\}$). The Wiener processes $dW_i(\tau)$, $i \in \{1,\dots,n\}$ are assumed to be uncorrelated and satisfy the following relations:

$$dW_i(\tau)\,dW_j(\tau) = \delta_{ij} d\tau, \quad (i,j) \in \{1,\dots,n\}.$$

For the SDE system (1) along with application of Ito calculus the FP equation of transition probability density of stochastic process [1, 4] is the following:

$$\frac{\partial K(x,t)}{\partial t} = \frac{1}{2}\sum_{i,j=1}^{n} \frac{\partial^2 \Sigma_{ij}(x) K(x,t)}{\partial x_i \partial x_j} - \sum_{i=1}^{n} \frac{\partial A_i(x) K(x,t)}{\partial x_i}. \tag{2}$$

Here $K(x,t)$, $A_i(x)$, $\Sigma_{ij}(x)$ depend on set of variables x_i, $(i,j) \in \{1,\dots,n\}$. The diffusion matrix $\mathbf{\Sigma}(x)$ is defined $\mathbf{B}(x)$:

$$\Sigma_{ij}(x) = \sum_{k=1}^{n} B_{ik} B_{jk}(x), \quad (i,j) \in \{1,\dots,n\}. \tag{3}$$

In the scheme of Stratonovich stochastic calculus for a number of intermediate schemas the FP equation differs by drift value $A_i(x)$, $i \in \{1,\dots,n\}$ [4]. Solution of (2) using path integral method is given in [12]. Transition probability density of equation variables (2) which corresponds to the system of SDE (1) is given by the following path integral [12] (see Appendix A):

$$\begin{aligned}
&K(x, x_0, t, t_0) = \int \tilde{\mathcal{D}}\nu(\tau) \exp\left(- \int_{t_0}^{t} L(x(\tau))\,d\tau \right) \prod_{i=1}^{n} \delta\left(x_i - x_{0i} - \int_{t_0}^{t} \nu_i(\tau)\,d\tau \right),\\
&L(x(\tau)) = L_0(x(\tau)) + U_0(x(\tau)),\\
&L_0(x(\tau)) = \frac{1}{2}\sum_{i,j=1}^{n} \big(\nu_i(\tau) - A_i^c(x(\tau))\big) \Sigma_{ij}^{-1}(x(\tau)) \big(\nu_j(\tau) - A_j^c(x(\tau))\big),\\
&U_0(x(\tau))) = \frac{1}{2}\sum_{i=1}^{n} \frac{\partial A_i(x(\tau))}{\partial x_i} - \frac{1}{8}\left(\sum_{ij}^{n} \frac{\partial^2 \Sigma_{i,j=1}(x(\tau))}{\partial x_i \partial x_j} + \sum_{i,j,k=1}^{n} \frac{\partial B_{jk}(x(\tau))}{\partial x_i} \frac{\partial B_{ik}(x(\tau))}{\partial x_j} \right).
\end{aligned} \tag{4}$$

Here $x_i(\tau)$, $A_i^c(x(\tau))$, $i \in \{1,\dots,n\}$ are defined in formulas (61) and (66) respectively, $x(\tau)$ denotes a set of variables $x_i(\tau)$, $i \in \{1,\dots,n\}$, an element of functional measure in (4) is equal to:

$$\tilde{\mathcal{D}}\nu(\tau) = \left(\prod_{\tau=t_0}^{t} \frac{1}{\det(B(x(\tau)))} \right) \prod_{k}^{n} \mathcal{D}\nu_k(\tau), \quad \mathcal{D}\nu_k(\tau) = \prod_{\tau=t_0}^{t} \sqrt{\frac{d\tau}{2\pi}}\,d\nu_k(\tau).$$

The path integral is defined in the velocity space $\nu_i(\tau)$ [7, 11, 25, 26], where connection with coordinates $x_i(\tau)$, $i \in \{1,\dots,n\}$ is defined in the last equality (4). For a one-dimensional use case the path integral (4) coincides with the expression given in [11].

3. Heston model

As it is known [13], the Heston model is defined by the following system of SDEs:

$$\begin{aligned} dS(\tau) &= rS(\tau)\,d\tau + S(\tau)\sqrt{V(\tau)}\,dW_s(\tau), \\ dV(\tau) &= \kappa(\theta - V(\tau))\,d\tau + \sigma\sqrt{V(\tau)}\,dW_v(\tau). \end{aligned} \tag{5}$$

The Heston model is an extension of Black–Scholes option pricing model in case of stochastic volatility. First equation models price dynamics $S(\tau)$ where volatility contains stochastic value $V(\tau)$ whose dynamics is given by second equation. Equation of price dynamics is a generalization of geometric Brownian motion, whereas the second equation is known as CIR stochastic process [13]. Wiener processes in equations (5) we consider to be correlated with correlation matrix $\mathbf{R}_H$

$$\mathbf{R}_H d\tau = \begin{pmatrix} dW_s(\tau) \\ dW_v(\tau) \end{pmatrix} \cdot \begin{pmatrix} dW_s(\tau)\, dW_V(\tau) \end{pmatrix} = \begin{pmatrix} 1 & \rho \\ \rho & 1 \end{pmatrix} d\tau.$$

As we already noted, the solutions for option price in Heston model [13] is obtained by substituting characteristic function of given structure into option price equation. Option price equation in Heston model was investigated by various methods [13], like already mentioned substitution of characteristic function, and by Laplace and Fourier transforms. The option price can be expressed as the discounted expected payoff $F_C(S) = \max(S - K, 0)$

$$C(t) = e^{-r(t-t_0)} \int_0^\infty \int_0^\infty K(S, S_0, V, V_0, t, t_0)\, F_C(S)\, dV\, dS, \tag{6}$$

where r is the interest rate, K is the strike price [13]. Based on (6) one can also obtain an option price equation which is an inverse Kolmogorov equation for $C(t)e^{r(t-t_0)}$ with initial condition $C(t)|_{t_0 \uparrow t} = F_C(S_0)$. Because pay function depends only on S, we will write (6) as

$$C(t) = e^{-r(t-t_0)} \int_0^\infty K(S, S_0, t, t_0)\, F_C(S)\, dS, \tag{7}$$

where $K(S, S_0, t, t_0)$ is the transition probability density for stochastic process related to $S(\tau)$, $\tau \in [t_0, t]$.

As it was already denoted, the FP equation for Heston model was solved in work [14], where the Laplace and Fourier transforms where used over variables S and V respectively, and also in [12] using path integral method.

3.1. Heston model I

In the classic Heston model the risk-free interest rate r is considered to be a constant. The extended Heston model takes into account stochastic dynamics of interest rate which we set using Vasicek stochastic equation:

$$\begin{aligned} dS(\tau) &= r(\tau)\,S(\tau)\,d\tau + S(\tau)\sqrt{V(\tau)}\,dW_s(\tau), \\ dV(\tau) &= \kappa(\theta - V(\tau))\,d\tau + \sigma_v\sqrt{V(\tau)}\,dW_v(\tau), \\ dr(\tau) &= \beta(\mu - r(\tau))\,d\tau + \sigma_r dW_r(\tau). \end{aligned} \tag{8}$$

In equation (1) we consider Wiener processes uncorrelated. System of equations for extended Heston model (8) we will write in the matrix form

$$\begin{pmatrix} dS(\tau) \\ dV(\tau) \\ dr(\tau) \end{pmatrix} = \mathbf{A}(\tau)\, d\tau + \mathbf{B}_0(\tau) \begin{pmatrix} dW_s(\tau) \\ dW_v(\tau) \\ dW_r(\tau) \end{pmatrix}, \tag{9}$$

where the following notations of drift vector and diffusion matrix of the model where used:

$$\mathbf{A}(\tau) = \begin{pmatrix} r(\tau)S(\tau) \\ \kappa(\theta - V(\tau)) \\ \beta(\mu - r(\tau)) \end{pmatrix}, \quad \mathbf{B}_0(\tau) = \begin{pmatrix} S(\tau)\sqrt{V(\tau)} & 0 & 0 \\ 0 & \sigma_v\sqrt{V(\tau)} & 0 \\ 0 & 0 & \sigma_r \end{pmatrix}. \tag{10}$$

We assume that in the Heston model the correlation takes place and process $dW_r(\tau)$ does not correlate with other Wiener processes. This way correlation matrix of Wiener processes $\mathbf{R}_H$ is the following

$$\mathbf{R}_H\, d\tau = \begin{pmatrix} dW_s \\ dW_v \\ dW_v \end{pmatrix} \cdot \begin{pmatrix} dW_s & dW_V & dW_r \end{pmatrix} = \begin{pmatrix} 1 & \rho & 0 \\ \rho & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} d\tau. \tag{11}$$

In order to switch to uncorrelated Wiener processes $\big(d\tilde{W}_s\, d\tilde{W}_v\, d\tilde{W}_r\big)$ we will define matrix L

$$\begin{pmatrix} dW_s \\ dW_v \\ dW_r \end{pmatrix} = \mathbf{L} \cdot \begin{pmatrix} d\tilde{W}_s \\ d\tilde{W}_v \\ d\tilde{W}_r \end{pmatrix}.$$

Substituting it into (11) and taking into account that correlation matrix of uncorrelated processes $\big(d\tilde{W}_s\, d\tilde{W}_v\, d\tilde{W}_r\big)$ is a unit matrix, we obtain the following solution

$$\mathbf{R}_H = \mathbf{L} \cdot \mathbf{L}^T.$$

For $\mathbf{L}$ we obtain

$$\mathbf{L} = \begin{pmatrix} 1 & 0 & 0 \\ \rho & \sqrt{1-\rho^2} & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

The respective diffusion matrix $\mathbf{B}$ is as the following

$$\mathbf{B} = \mathbf{B}_0 \mathbf{L} = \begin{pmatrix} S(\tau)\sqrt{V(\tau)} & 0 & 0 \\ \rho\sigma_v\sqrt{V(\tau)} & \sigma_v\sqrt{1-\rho^2}\sqrt{V(\tau)} & 0 \\ 0 & 0 & \sigma_r \end{pmatrix}. \tag{12}$$

3.2. Transition probability density of the model

Substituting drift vector (10) and diffusion matrix (12) into general formulas (4) we obtain

$$K(S, S_0, V, V_0, r, r_0, t, t_0) = \int \tilde{\mathcal{D}}\nu(\tau) \exp\left(-\int_{t_0}^{t} L(\tau))\, d\tau\right) \prod_{i \in s,r,v} \delta\left(x_i - x_{0i} - \int_{t_0}^{t} \nu_i(\tau) d\tau\right), \tag{13}$$

with following notations:

$$L(\tau) = L_0(\tau) + U_0(\tau),$$

$$L_0(\tau) = \frac{1}{2}\frac{(\nu_s(\tau) - a_s(\tau)S(\tau))^2}{(1-\rho^2)S(\tau)^2 V(\tau)} + \frac{1}{2}\frac{(\nu_r(\tau) - a_r(\tau))^2}{\sigma_r^2} + \frac{1}{2}\frac{(\nu_v(\tau) - a_v(\tau))^2}{\sigma_v^2 V(\tau)}, \tag{14}$$

$$U_0(\tau) = \frac{1}{2} r(\tau) - \frac{\sigma_v^2}{32 V(\tau)} - \frac{3}{8} V(\tau) - \frac{1}{2}(\beta + \kappa) - \frac{1}{4}\rho\,\sigma_v, \tag{15}$$

and also:

$$\begin{aligned} a_s(\tau) &= \frac{\rho\nu_v(\tau)}{\sigma_v} + r(\tau) + \frac{1}{2}\left(\frac{2\kappa\rho}{\sigma_v} - (2-\rho^2)\right) V(\tau) - \frac{1}{2}\alpha\rho\,\sigma_v, \\ a_r(\tau) &= \beta\big(\mu - r(\tau)\big), \\ a_v(\tau) &= \frac{1}{2}\big((\alpha-1)\sigma_v^2 - (2\kappa + \rho\,\sigma_v)V(\tau)\big). \end{aligned} \tag{16}$$

Here $\alpha = \frac{2\theta\kappa}{\sigma_v^2}$ denotes Feller parameter for stochastic process of volatility [13]. In the formulas described, the following reassignments were used:

$$x_s(\tau) \to S(\tau), \quad x_r(\tau) \to r(\tau), \quad x_v(\tau) \to V(\tau).$$

Element of functional measure (13) is defined by the following expression:

$$\begin{aligned} \tilde{\mathcal{D}}\nu &= \left(\prod_{\tau=t_0}^{t} \frac{1}{\det(\mathbf{B}(\tau))}\right) \prod_{k\in\{s,r,v\}} \mathcal{D}\nu_k, \quad \mathcal{D}\nu_k = \prod_{\tau=t_0}^{t} \sqrt{\frac{d\tau}{2\pi}}\, d\nu_k(\tau), \quad k \in \{s,r,v\}, \\ \det(\mathbf{B}(\tau)) &= \sqrt{1-\rho^2}\,\sigma_r\sigma_v S(\tau)V(\tau). \end{aligned} \tag{17}$$

Path integrals in (13) are calculated consecutively, first over variable $S(\tau)$. According to the structure of first term in (14) we will perform variable substitution in path integral $\nu_s(\tau) \to q_s(\tau)$

$$\frac{\nu_s(\tau) - a_s(\tau)S(\tau)}{\sqrt{1-\rho^2}S(\tau)\sqrt{V(\tau)}} = q_s(\tau). \tag{18}$$

Let consider a first order differential equation (18) for function $S(\tau)$ ($\dot{S}(\tau) = \nu_s(\tau)$) with initial condition $S(t) = S$ and arbitrary functions $V(\tau)$, $q_s(\tau)$. It is solution we will write in the following form

$$S(\tau) = \exp\left\{-\int_\tau^t a_s(\tau_1)\,d\tau_1 - \sqrt{1-\rho^2}\int_\tau^t \sqrt{V(\tau_1)}\,q_s(\tau_1)\,d\tau_1\right\} S.$$

For corresponding δ-function in formula (4) we obtain

$$\delta\biggl(S - S_0 - \int_{t_0}^t \nu_s(\tau)\,d\tau\biggr) \to \delta\biggl(\exp\left\{-\int_{t_0}^t a_s(\tau)\,d\tau - \sqrt{1-\rho^2}\int_{t_0}^t \sqrt{V(\tau)}\,q_s(\tau)\,d\tau\right\} S - S_0\biggr). \tag{19}$$

Jacobian of variable substitution (18) is calculated using approach from Appendix C (see also [12])

$$J_S = \left\|\frac{\delta\nu_s(\tau)}{\delta q_s(\tau')}\right\| = \biggl(\prod_{\tau=t_0}^t S(\tau)\sqrt{V(\tau)}\biggr)\sqrt{\frac{S_0}{S}}. \tag{20}$$

First multiplier of the Jacobian (20) we cancel out with (17). In order to calculate path integral over $q_s(\tau), \tau \in [t_0, t]$ we use Fourier transform of δ-function (19) which is described in Appendix D. As a result for path integral of variable $q_s(\tau)$, $\tau \in [t_0, t]$, we obtain

$$\begin{aligned}
&\sim \frac{1}{\sqrt{S_0 S}}\int \mathcal{D}q_s(\tau)\exp\biggl(-\frac{1}{2}\int_{t_0}^t q_s(\tau)^2 d\tau\biggr) \\
&\quad\times \frac{1}{2\pi}\int_{-\infty}^{\infty}\exp\biggl(-ik\ln\frac{S_0}{S}\biggr)\exp\left\{-ik\biggl(\int_{t_0}^t a_s(\tau)\,d\tau + \sqrt{1-\rho^2}\int_{t_0}^t q_s(\tau)\sqrt{V(\tau)}\,d\tau\biggr)\right\}dk \\
&\quad= \frac{1}{\sqrt{S_0 S}}\frac{1}{2\pi}\int_{-\infty}^{\infty}\exp\left\{-ik\biggl(\ln\frac{S_0}{S} + \int_{t_0}^t a_s(\tau)\,d\tau\biggr)\right\}\exp\left\{-\frac{k^2}{2}(1-\rho^2)\int_{t_0}^t V(\tau)\,d\tau\right\}dk.
\end{aligned} \tag{21}$$

As it is noted in (7), the transition probability density for the stochastic process $S(\tau)$ is important for application. Let us integrate over variable r, which will reduce to the following substitution in all expressions, given the presence of delta function $\delta\bigl(r - r_0 - \int_{t_0}^t \nu_r(\tau)\,d\tau\bigr)$

$$r(\tau) = r - \int_\tau^t \nu_r(\tau_1)\,d\tau_1 \to r_0 + \int_{t_0}^\tau \nu_r(\tau_1)\,d\tau_1.$$

In order to calculate path integral over $\nu_r(\tau)$ let us perform variable substitution in path integral $\nu_r(\tau) \to q_r(\tau)$ using formula

$$\frac{\nu_r(\tau) - a_r(\tau)}{\sigma_r} = q_r(\tau). \tag{22}$$

Let us find solution to the first order differential equation (22) for $r(\tau)$ ($\nu_r(\tau) = \dot{r}(\tau)$) with initial condition $r(t_0) = r_0$. We obtain the following

$$r(\tau) = \mu + (r_0 - \mu)\,e^{-\beta(\tau - t_0)} + \sigma_r\int_{t_0}^\tau e^{-\beta(\tau-\tau_1)}q_r(\tau_1)\,d\tau_1. \tag{23}$$

Jacobian of variable substitution (22) is equal to

$$J_r = \biggl(\prod_{\tau=t_0}^t \sigma_r\biggr)e^{-\frac{1}{2}\beta(t-t_0)}. \tag{24}$$

First multiplier in (24) will cancel out with the identical multiplier in (17). For path integral over variable $q_r(\tau)$, $\tau \in [t_0, t]$ we obtain

$$\sim e^{-\frac{1}{2}\beta(t-t_0)}\int \mathcal{D}q_r(\tau)\exp\left\{-\frac{1}{2}\int_{t_0}^t q_r(\tau)^2 d\tau\right\}\exp\left\{-\biggl(\frac{1}{2}+ik\biggr)\int_{t_0}^t r(\tau)\,d\tau\right\}. \tag{25}$$

Instead of $r(\tau)$ in (25) let us substitute solution of (23) and after calculation of the path integral we obtain

$$\sim \exp\Big(-\frac{1}{2}\beta(t-t_0)\Big)\exp\Big\{-\Big(\frac{1}{2}+ik\Big)\int_{t_0}^{t}\langle r(\tau)\rangle\,d\tau\Big\}\exp\Big\{\frac{1}{2}\Big(\frac{1}{2}+ik\Big)^2\int_{t_0}^{t}B(t-\tau)^2 d\tau\Big\}. \quad (26)$$

The following notation are used:

$$\langle r(\tau)\rangle = \mu + (r_0-\mu)\,e^{-\beta(\tau-t_0)}, \quad B(\tau) = \frac{\sigma_r}{\beta}\big(1-e^{-\beta\tau}\big).$$

It is impossible to perform similar variable substitution in path integral over $q_v(\tau)$ (the third term in (14)). Since differential equation for $V(\tau)$

$$\frac{\dot V(\tau)-a_v(\tau)}{\sigma_v\sqrt{V(\tau)}} = q_v(\tau)$$

belongs to the Abel equation, for which no solution exist in general form [27]. Therefore, let us perform a number of transformations. In particular, let us sum terms from (14), (15) and in exponent (21)

$$\Delta L(\tau) = \frac{1}{2}\frac{(\nu_v(\tau)-a_v(\tau))^2}{\sigma_v^2 V(\tau)} + \tilde U_0(\tau) + ik\,\tilde a_s(\tau) + \frac{1}{2}k^2(1-\rho^2)\,V(\tau). \quad (27)$$

Here values $\tilde U_0(\tau)$, $\tilde a_s(\tau)$ are defined by formulas (15), (16) without terms from $r(\tau)$.

First term in expression $\Delta L(\tau)$ we will write in the form

$$\frac{1}{2}\frac{(\nu_v(\tau)-a_v(\tau))^2}{\sigma_v^2 V(\tau)} = \frac{1}{2}\frac{\nu_v(\tau)^2}{\sigma_v^2 V(\tau)} - \frac{\nu_v(\tau)\,a_v(\tau)}{\sigma_v^2 V(\tau)} + \frac{1}{2}\frac{a_v(\tau)^2}{\sigma_v^2 V(\tau)}. \quad (28)$$

For terms $\sim \nu_v(\tau)(\nu_v(\tau)=\dot V(\tau))$ in (27) we obtain

$$\exp\Big\{\int_{t_0}^{t}\Big(\frac{a_v(\tau)}{\sigma_v^2 V(\tau)} - ik\frac{\rho}{\sigma_v}\Big)\nu_v(\tau)\,d\tau\Big\} = \exp\Big\{\int_{t_0}^{t}\Big(\frac{a_v(\tau)}{\sigma_v^2 V(\tau)} - ik\frac{\rho}{\sigma_v}\Big)dV(\tau)\Big\}$$

$$= \exp\Big\{-\frac{(V-V_0)(2\kappa+(1+2ik)\rho\sigma_v)}{2\sigma_v^2}\Big\}\Big(\frac{V}{V_0}\Big)^{-\frac{1}{2}(1-\alpha)}. \quad (29)$$

According to the first term (28) let us perform variable substitution

$$\frac{\nu_v(\tau)}{\sigma\sqrt{V(\tau)}} = q_v(\tau). \quad (30)$$

Differential equation (30) we will solve for function $V(\tau)$ and initial condition $V(t)=V$:

$$\frac{\dot V(\tau)}{\sigma_v\sqrt{V(\tau)}} = \dot z_v(\tau), \quad z_v(\tau) = z_v - \int_{\tau}^{t} q_v(\tau_1)\,d\tau_1,$$

we obtain

$$V(\tau) = \frac{1}{4}\sigma_v^2 z_v(\tau)^2. \quad (31)$$

Jacobian for variable substitution (30) that we calculate using approach described in Appendix C is equal to

$$J_V = \Big\|\frac{\delta\nu_v(\tau)}{\delta q_v(\tau')}\Big\| = \Big(\prod_{\tau=t_0}^{t}\sigma_v\sqrt{V(\tau)}\Big)\sqrt[4]{\frac{V_0}{V}}. \quad (32)$$

First multiplier in the right side (32) cancels out with the respective multiplier in (17). Substituting (31) in other terms (27) we obtain

$$-\int_{t_0}^{t}\Delta\tilde L(\tau)\,d\tau = \frac{1}{2}(\beta+\alpha\gamma)(t-t_0) - \frac{1}{2}\int_{t_0}^{t}\dot z_v^2\,d\tau - \frac{\lambda^2-\frac{1}{4}}{2}\int_{t_0}^{t}\frac{1}{z_v(\tau)^2}\,d\tau - \frac{\omega^2}{2}\int_{t_0}^{t}z_v(\tau)^2 d\tau. \quad (33)$$

Following notations were introduced:

$$\gamma = \kappa + \Big(\frac{1}{2}+ik\Big)\rho\sigma_v, \quad \lambda = \alpha-1, \quad \omega = \frac{1}{2}\sqrt{\gamma^2 - \Big(\frac{1}{2}+ik\Big)\Big(\frac{3}{2}+ik\Big)\sigma_v^2}. \quad (34)$$

Path integral for expression in (33) is given in Appendix E. As a result we obtain

$$\sim \int \mathcal{D}q_v(\tau) \exp\left(- \int_{t_0}^{t} \Delta\tilde{L}(\tau)\, d\tau \right)$$
$$= \frac{1}{2}\left(\frac{V}{V_0}\right)^{\frac{1}{4}} \exp\left\{\frac{1}{2}(\beta+\alpha\gamma)(t-t_0)\right\} \exp\left\{-\frac{2}{\sigma_v^2}(V+V_0)\,\omega\coth(\omega(t-t_0))\right\}$$
$$\times \frac{\omega}{\sinh\left(\omega(t-t_0)\right)} I_{\alpha-1}\left(\frac{4\sqrt{V_0V}\omega}{\sigma_v^2\sinh\left(\omega(t-t_0)\right)}\right). \quad (35)$$

As in case of ordinary Heston model we transform to stochastic variable $x = \ln\frac{S}{S_0}$. For multipliers S_0, S in (21) taking into account Jacobian of variable substitution we obtain

$$\frac{1}{\sqrt{S_0 S}}\frac{dS}{dx} = e^{\frac{x}{2}}. \quad (36)$$

Multiplier (36) we will eliminate using transform $k \to k + \frac{i}{2}$ in Fourier integral (21). We will also perform respective transformation in expressions (26), (29), (34), (35). Let us combine given results and write down transition probability density

$$K(x,V,t) = \frac{2}{\sigma_v^2}\left(\frac{V}{V_0}\right)^{\frac{1}{2}(\alpha-1)} \frac{1}{2\pi}\int_{-\infty}^{\infty} \exp\left\{ik\left(x - \int_{t_0}^{t}\langle r(\tau)\rangle\, d\tau\right)\right\}$$
$$\times \exp\left\{\frac{1}{2}\alpha\gamma(t-t_0)\right\}\exp\left\{-\frac{\gamma(V-V_0)}{\sigma_v^2}\right\}\exp\left\{-\frac{2(V+V_0)\omega\coth(\omega(t-t_0))}{\sigma_v^2}\right\}$$
$$\times \exp\left\{-\frac{1}{2}k^2\int_{t_0}^{t}B(t-\tau)^2 d\tau\right\}\frac{\omega}{\sinh\left(\omega(t-t_0)\right)} I_{\alpha-1}\left(\frac{4\sqrt{V_0V}\omega}{\sigma_v^2\sinh\left(\omega(t-t_0)\right)}\right).$$

Integrating it with respect to V, we obtain the transition probability density for variable x

$$K(x,t) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\exp\left\{ik\left(x-\int_{t_0}^{t}\langle r(\tau)\rangle\, d\tau\right)\right\}\exp\left(\frac{1}{2}\alpha\gamma(t-t_0)\right)\exp\left\{-\frac{k(k-i)V_0}{\gamma+2\omega\coth(\omega(t-t_0))}\right\}$$
$$\times \exp\left(-\frac{1}{2}k^2\int_{t_0}^{t}B(t-\tau)^2 d\tau\right)\left(\cosh(\omega(t-t_0)) + \frac{\gamma\sinh(\omega(t-t_0))}{2\omega}\right)^{-\alpha} dk. \quad (37)$$

Following is denoted:

$$\gamma = \kappa + ik\rho\sigma_v, \quad \lambda = \alpha - 1, \quad \omega = \frac{1}{2}\sqrt{\gamma^2 + k(k-i)\sigma_v^2}. \quad (38)$$

This way (37) gives solution for transition probability density for variable x in extended Heston model. From formula (37) one can see that it defines enclosed expression for characteristic function of the model of variable x, since integral over k has no analytic solution. From formula (37) it also follows that $K(x,t)$ is normalized and at the limit $\beta \to 0$, $\sigma_r \to 0$ we obtain transition probability density for Heston model [12]

$$K(x,t) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\exp\left\{ik\left(x - r_0(t-t_0)\right)\right\}\exp\left(\frac{1}{2}\alpha\gamma(t-t_0)\right)\exp\left(-\frac{k(k-i)V_0}{\gamma+2\omega\coth(\omega(t-t_0))}\right)$$
$$\times\left(\cosh(\omega(t-t_0)) + \frac{\gamma\sinh(\omega(t-t_0))}{2\omega}\right)^{-\alpha} dk. \quad (39)$$

3.3. Heston model II

Let us consider extension of the Heston model where stochastic interest rate is described by CIR process:

$$\begin{aligned} dS(\tau) &= r(\tau)\, S(\tau)\, d\tau + S(\tau)\sqrt{V(\tau)}\, dW_s(\tau), \\ dV(\tau) &= \kappa(\theta - V(\tau))\, d\tau + \sigma_v\sqrt{V(\tau)}\, dW_v(\tau), \\ dr(\tau) &= \beta(\mu - r(\tau))\, d\tau + \sigma_r\sqrt{r(\tau)}\, dW_r(\tau). \end{aligned} \quad (40)$$

Both the volatility $V(\tau)$ and the interest rate dynamics $r(\tau)$ are given by stochastic processes. Correlation matrix is given as in the previous case (11). Based on (40), we will write drift vector and diffusion matrix of the model:

$$\mathbf{A}(\tau) = \begin{pmatrix} r(\tau)S(\tau) \\ \kappa(\theta - V(\tau)) \\ \beta(\mu - r(\tau)) \end{pmatrix}, \quad \mathbf{B}(\tau) = \begin{pmatrix} S(\tau)\sqrt{V(\tau)} & 0 & 0 \\ \rho\sigma_v\sqrt{V(\tau)} & \sqrt{1-\rho^2}\sigma_v\sqrt{V(\tau)} & 0 \\ 0 & 0 & \sigma_r\sqrt{r(\tau)} \end{pmatrix}.$$

For the given model, we obtain the path integral (13) with values:

$$L(\tau) = L_0(\tau) + U_0(\tau),$$

$$L_0(\tau) = \frac{1}{2}\frac{(\nu_s(\tau) - a_s(\tau)S(\tau))^2}{(1-\rho^2)S(\tau)^2V(\tau)} + \frac{1}{2}\frac{(\nu_r(\tau) - a_r(\tau))^2}{\sigma_r^2 r(\tau)} + \frac{1}{2}\frac{(\nu_v(\tau) - a_v(\tau))^2}{\sigma_v^2 V(\tau)},$$

$$U_0(\tau) = -\frac{\sigma_r^2}{32\,r(\tau)} + \frac{1}{2}r(\tau) - \frac{\sigma_v^2}{32V(\tau)} - \frac{3}{8}V(\tau) - \frac{1}{2}(\beta+\kappa) - \frac{1}{4}\rho\,\sigma_v. \tag{41}$$

The following notation is used:

$$a_s(\tau) = \frac{\rho\nu_v(\tau)}{\sigma_v} + r(\tau) + \frac{1}{2}\left(\frac{2\kappa\rho}{\sigma_v} - (2-\rho^2)\right)V(\tau) - \frac{1}{2}\alpha\rho\,\sigma_v,$$

$$a_r(\tau) = \frac{1}{2}(\alpha_r - 1)\,\sigma_r^2 - \beta r(\tau),$$

$$a_v(\tau) = \frac{1}{2}\big((\alpha - 1)\,\sigma_v^2 - (2\kappa + \rho\,\sigma_v)V(\tau)\big),$$

and also Feller parameter $\alpha_r = \frac{2\beta\mu}{\sigma_r^2}$ for stochastic process of interest rate.

The path integrals are evaluated in an analogous way to the previous case. Integral for process $S(\tau)$ is reduced to Gaussian path integral as in (21). Path integral for processes $r(\tau)$, $V(\tau)$ are reduced to integrals of radial oscillator type as in (33). We do not provide the calculation details here, but only the resulting expressions for transition probability density for stochastic variable $x = \ln\frac{S}{S_0}$

$$K(x,t) = \exp\left\{\frac{1}{2}\alpha_r\beta(t-t_0)\right\}\frac{1}{2\pi}\int_{-\infty}^{\infty}\exp\left\{ik\left(x - \frac{2\,r_0}{\beta + 2\omega_r\coth(\omega_r(t-t_0))}\right)\right\}\exp\left\{\frac{1}{2}\alpha\gamma(t-t_0)\right\}$$
$$\times\exp\left\{-\frac{k(k-i)V_0}{\gamma + 2\,\omega\coth(\omega(t-t_0))}\right\}\left(\cosh(\omega_r(t-t_0)) + \frac{\beta\sinh(\omega_r(t-t_0))}{2\,\omega_r}\right)^{-\alpha_r}$$
$$\times\left(\cosh(\omega(t-t_0)) + \frac{\gamma\sinh(\omega(t-t_0))}{2\,\omega}\right)^{-\alpha}. \tag{42}$$

Here γ, λ, ω are given by (38) and also $\omega_r = \frac{1}{2}\sqrt{\beta^2 + 2ik\,\sigma_r^2}$.

In case of constant interest rate the expression (42) is consistent with the transition probability density of Heston model (39).

3.4. Case of correlation for stochastic process of interest rate

Let us consider additional correlation between processes $dW_s(\tau)$, $dW_r(\tau)$ for Heston model I (8). As shown in [15], within the characteristic-function approach the model has no analytic solution in this case. Correlation matrix of Wiener process is the following

$$\mathbf{R}_H = \begin{pmatrix} 1 & \rho & \rho_r \\ \rho & 1 & 0 \\ \rho_r & 0 & 1 \end{pmatrix}.$$

Respectively matrix L of transition to independent Wiener processes equals to

$$\mathbf{L} = \begin{pmatrix} 1 & 0 & 0 \\ \rho & \sqrt{1-\rho^2} & 0 \\ \rho_r & -\frac{\rho\rho_r}{\sqrt{1-\rho^2}} & \sqrt{\frac{1-\rho^2-\rho_r^2}{1-\rho^2}} \end{pmatrix}.$$

The diffusion matrix of model Heston I is the following

$$\mathbf{B}(\tau) = \begin{pmatrix} S(\tau)\sqrt{V(\tau)} & 0 & 0 \\ \rho\sigma_v\sqrt{V(\tau)} & \sqrt{1-\rho^2}\sigma_v\sqrt{V(\tau)} & 0 \\ \rho_r\sigma_r & -\frac{\rho\rho_r\sigma_r}{\sqrt{1-\rho^2}} & \sqrt{\frac{1-\rho^2-\rho_r^2}{1-\rho^2}}\sigma_r \end{pmatrix}.$$

Drift vector $\mathbf{A}(\tau)$ is given by formula (10). As result, we obtain the path integral (13) with the following values:

$$\begin{aligned} L(\tau) &= L_0(\tau) + U_0(\tau), \\ L_0(\tau) &= \frac{1}{2}\frac{(\nu_s(\tau) - a_s(\tau)S(\tau))^2}{(1-\rho^2)S(\tau)^2V(\tau)} + \frac{1}{2}\frac{(\nu_r(\tau) - a_r(\tau))^2}{\sigma_r^2} + \frac{1}{2}\frac{(\nu_v(\tau) - a_v(\tau))^2}{\sigma_v^2 V(\tau)}, \\ U_0(\tau) &= \frac{1}{2}r(\tau) - \frac{\sigma_v^2}{32V(\tau)} - \frac{3}{8}V(\tau) - \frac{1}{2}(\beta+\kappa) - \frac{1}{4}\rho\,\sigma_v. \end{aligned}$$

The following notation are used:

$$\begin{aligned} a_s(\tau) &= \frac{\rho\nu_v(\tau)}{\sigma_v} + r(\tau) + \frac{\rho_r}{\sigma_r}\big(\nu_r(\tau) + \beta(r(\tau)-\mu)\big)\sqrt{V(\tau)} + \left(\frac{\kappa\rho}{\sigma_v} - \frac{1}{2}\big(2-\rho^2-\rho_r^2\big)\right)V(\tau) - \frac{1}{2}\alpha\rho\sigma_v, \\ a_r(\tau) &= \beta\big(\mu - r(\tau)\big) - \frac{1}{2}\rho_r\sigma_r\sqrt{V(\tau)}, \qquad (43) \\ a_v(\tau) &= \frac{1}{2}\big((\alpha-1)\sigma_v^2 - (2\kappa+\rho\,\sigma_v)V(\tau)\big). \end{aligned}$$

Path integrals that correspond to stochastic processes $S(\tau)$, $r(\tau)$ are calculated similarly to (21), (25) and are reduced to Gaussian integrals. However because of terms $\sim\sqrt{V(\tau)}$ in values $a_s(\tau)$, $a_r(\tau)$ in (43) given model has no analytic solution. Indeed path integrals for stochastic process $V(\tau)$ in formula (33) contain terms like

$$\sim \int_{t_0}^{t} f(\tau)\sqrt{V(\tau)}\,d\tau.$$

It is known that such functions have no analytical solution.

Also extended Heston model II with specified additional correlation has no analytic solution. This follows from the fact that path integral for processes $r(\tau)$, $V(\tau)$ contain term

$$\sim \int_{t_0}^{t} \sqrt{r(\tau)}\sqrt{V(\tau)}\,d\tau,$$

and cannot be calculated analytically. This way in case of specified correlation one needs to consider approximate methods of solving path integrals.

Let us point out that terms with multipliers $\sim\sqrt{r(\tau)}$, $\sim\sqrt{V(\tau)}$ are contained in partial differential equations for characteristic function [15]. In the work it is proposed a simple approximation for solution of the given equation, mainly substituting them with an average value:

$$\sqrt{r(\tau)} \to \big\langle\sqrt{r(\tau)}\big\rangle, \quad \sqrt{V(\tau)} \to \big\langle\sqrt{V(\tau)}\big\rangle, \quad \sqrt{V(\tau)}\sqrt{r(\tau)} \to \big\langle\sqrt{V(\tau)}\sqrt{r(\tau)}\big\rangle.$$

3.5. Option price formula

In case of stochastic interest rate, the option price formula is the following (see (6))

$$C(t) = \left\langle \exp\left(-\int_{t_0}^{t} r(\tau)\,d\tau\right) F_C(S)\right\rangle, \qquad (44)$$

where $\langle\ldots\rangle$ denotes average over all stochastic processes. Option price (44) we will express in path integral

$$C(t) = \int\left(\int \tilde{\mathcal{D}}\nu(\tau)\exp\left(-\int_{t_0}^{t}\tilde{L}(\tau)\right)d\tau\right)\prod_{i\in s,r,v}\delta\left(x_i - x_{0i} - \int_{t_0}^{t}\nu_i(\tau)\,d\tau\right)(S-K)^+ dS\,dr\,dV. \quad (45)$$

Here $\tilde{L}(\tau))$ in the considered models (13) and (40) will be defined by respective values with only one variation, which is that in formulas (15) and (41) the following substitution should be introduced

$$U_0(\tau) \to U_0(\tau) + r(\tau). \tag{46}$$

As a result for path integral (45) we obtain the following

$$C(t) = \int_{\ln\frac{K}{S_0}}^{\infty} \left(\frac{1}{2\pi}\int_{-\infty}^{\infty} F(k)\, e^{ikx}\, dk\right)\big(S_0 e^x - K\big)\, dx. \tag{47}$$

Here $F(k)$ is determined by a fairly simple way based on transition probability densities (37) and (42), and substitution (46).

We now transform (47) by changing the order of integration. For that we will perform shift in the complex plane $\{k_c,\, k_s\}$, $k = k_s + i\, k_s$. This means that in the first term

$$S_0 \frac{1}{2\pi}\int_{-\infty}^{\infty} F(k) \bigg(\int_{\ln\frac{K}{S_0}}^{\infty} e^{ikx} e^{x} dx\bigg) dk - K\frac{1}{2\pi}\int_{-\infty}^{\infty} F(k)\bigg(\int_{\ln\frac{K}{S_0}}^{\infty} e^{ikx} dx\bigg) dk$$

we will perform shift $k \to k + i\,(1+\varepsilon)$, and in the second one — $k \to k + i\varepsilon$ ($\varepsilon > 0$). Also, using the know formula

$$\frac{1}{k + i\varepsilon} = \mathcal{P}\frac{1}{k} - i\,\pi\,\delta(k),$$

after integrating over x we obtain

$$C(t) = S_0 P_1 - K P_2, \tag{48}$$

where the following notations are used

$$\begin{aligned} P_1 &= \frac{1}{2}F(i) - \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{F(k+i)}{i\,k}\, e^{ik\ln\frac{K}{S_0}} dk, \\ P_2 &= \frac{1}{2}F(0) - \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{F(k)}{i\,k}\, e^{ik\ln\frac{K}{S_0}} dk. \end{aligned} \tag{49}$$

Integrals in (49) are considered in the sense of principal value.

For Heston I model, we compute the value $F_I(k)$ based on (37) and substitution (46)

$$\begin{aligned} F_I(k) = \exp\left(-(1+ik)\int_{t_0}^{t} \langle r(\tau)\rangle\, d\tau\right) \exp\left(\frac{1}{2}\alpha\gamma(t-t_0)\right) \exp\left(-\frac{k(k-i)V_0}{\gamma + 2\omega\coth(\omega(t-t_0))}\right) \\ \times \exp\left\{\frac{1}{2}(1+ik)^2 \int_{t_0}^{t} B(t-\tau)^2 d\tau\right\} \left(\cosh(\omega(t-t_0)) + \frac{\gamma\sinh(\omega(t-t_0))}{2\omega}\right)^{-\alpha}. \end{aligned}$$

Correspondingly, we obtain the value $F_{II}(k)$ for the Heston II model

$$\begin{aligned} F_{II}(k) = \exp\left(\frac{1}{2}(\alpha_r\beta + \alpha\gamma)(t-t_0)\right) \exp\left(-\frac{2(1+ik)\, r_0}{\beta + 2\omega_r\coth(\omega_r(t-t_0))}\right) \\ \times \exp\left(-\frac{k(k-i)V_0}{\gamma + 2\,\omega\coth(\omega(t-t_0))}\right) \left(\cosh(\omega_r(t-t_0)) + \frac{\beta\sinh(\omega_r(t-t_0))}{2\,\omega_r}\right)^{-\alpha_r} \\ \times \left(\cosh(\omega(t-t_0)) + \frac{\gamma\sinh(\omega(t-t_0))}{2\,\omega}\right)^{-\alpha}. \end{aligned}$$

Here ω_r is given by expression $\omega_r = \frac{1}{2}\sqrt{\beta^2 + 2(1+ik)\sigma_r^2}$.

For the Heston I model we obtain:

$$F_I(i) = 1, \quad F_I(0) = \exp\left(-\int_{t_0}^{t} \langle r(\tau)\rangle\, d\tau\right) \exp\left(\frac{1}{2}\int_{t_0}^{t} B(t-\tau)^2 d\tau\right).$$

Expression $F_I(0)$ coincides with the time structure of interest rate in Vasicek model [3, 24]. Likewise, for the Heston II model we find the following:

$$\begin{aligned} F_{II}(i) = 1, \qquad F_{II}(0) = \exp\left(\frac{1}{2}\alpha_r\beta(t-t_0)\right) \exp\left(-\frac{2\, r_0}{\beta + 2\omega_0\coth(\omega_0(t-t_0))}\right) \\ \times \left(\cosh(\omega_0(t-t_0)) + \frac{\beta\sinh(\omega_0(t-t_0))}{2\,\omega_0}\right)^{-\alpha_r}, \quad \omega_0 = \tfrac{1}{2}\sqrt{\beta^2 + 2\sigma_r^2}. \end{aligned}$$

Value $F_{II}(0)$ determines time structure of interest rate in CIR model [3]. In case of constant interest rate, values $F_I(0) = F_{II}(0) = e^{-r_0(t-t_0)}$ set a discount multiplier and formulas (48), (49) coincide with the standard Heston model [13].

4. Conclusions

The path integral method has been applied to the multi-dimensional Fokker–Planck (FP) equation corresponding to a system of stochastic differential equations (SDEs). A path–integral representation of the transition probability density for the underlying stochastic variables has been obtained, which is equivalent to the solution of the associated SDE system. The obtained path integrals were then used to analyse extended Heston models in which the short rate $r(\tau)$ is governed by an additional stochastic equation: by a Vasicek process in the first case and by a CIR process in the second.

Closed-form expressions have been derived for these extended Heston models in the absence of correlation between the Wiener processes driving $S(\tau)$ and $r(\tau)$, and an option–pricing formula with the standard Heston-type structure has been obtained. In the presence of additional correlation involving the interest-rate Wiener process, the models no longer admit analytic solutions, so approximate methods must be employed to evaluate the corresponding path integrals. These and related issues will be addressed in future work.

A. Appendix

We obtain solution to the FP equation (2) using path integral method with slightly modified approach from [12]. Let us write the FP equation (2) using operator form

$$\frac{\partial K(x,t)}{\partial t} = -\mathcal{H}K(x,t),$$

where $\mathcal{H}$ is the operator of FP equation

$$\mathcal{H} = -\frac{1}{2}\sum_{i,j=1}^{n}\frac{\partial^2}{\partial x_i \partial x_j}\Sigma_{ij}(x) + \sum_{i=1}^{n}\frac{\partial}{\partial x_i}A_i(x). \tag{50}$$

Solution for the transition probability density of stochastic variables we will build based on operator exponent

$$K(x,x_0,t,t_0) = e^{-(t-t_0)\mathcal{H}}\prod_{i=1}^{n}\delta(x_i - x_{0i}). \tag{51}$$

Formula (51) sets the solution of Cauchy problem for FP equation with the following initial condition

$$K(x,x_0,t,t_0)\big|_{t\downarrow t_0} = \prod_{i=1}^{n}\delta(x_i - x_{0i}).$$

Next let us write operator (50) in identical form

$$\mathcal{H} = -\frac{1}{2}\sum_{k=1}^{n}\hat{\mathcal{P}}_k^2 + U(x). \tag{52}$$

The following operator notations are used

$$\hat{\mathcal{P}}_k = \sum_{i=1}^{n} B_{ik}(x)\frac{\partial}{\partial x_i} - p_k(x), \quad k \in \{1,\dots,n\}. \tag{53}$$

Values $U(x)$ and $p_k(x)$, $k \in \{1,\dots,n\}$ we will determine using operators (50) and (52) equality condition. After a number of transformations we obtain a system of equation for determining $p_k(x)$, $k \in \{1,\dots,n\}$:

$$\sum_{k}^{n} B_{ik}(x)\left(p_k(x) + \sum_{j}^{n}\frac{\partial B_{jk}(x)}{\partial x_j}\right) = A_i(x) - \frac{1}{2}\sum_{k,j=1}^{n}\frac{\partial B_{ik}(x)}{\partial x_j}B_{jk}(x), \quad i \in \{1,\dots,n\}, \tag{54}$$

and also expression for $U(x)$:

$$\begin{aligned} U(x) &= \frac{1}{2}\sum_{k=1}^{n}\tilde{p}_k(x)^2 + \frac{1}{2}\sum_{k,i=1}^{n}\frac{\partial \tilde{p}_k(x)}{\partial x_i}B_{ik}(x), \\ \tilde{p}_k(x) &= p_k(x) + \sum_{i=1}^{n}\frac{\partial B_{ik}(x)}{\partial x_i}, \quad k \in \{1,\dots,n\}. \end{aligned} \tag{55}$$

From the system of equations (54) it follows that representation (52) is correct if matrix $B_{ik}(x)$, $(i,k) \in \{1,\ldots,n\}$ is inversible and the system of equations (53) is unambiguously solvable for p_k, $k \in \{1,\ldots,n\}$. For operator exponent in (51) taking form (52) into account we will apply Gaussian path integral [11, 12, 25, 26]

$$e^{-(t-t_0)\mathcal{H}} = \int \mathcal{D}q(\tau) \exp\left(-\frac{1}{2}\sum_k^n \int_{t_0}^t q_k^2(\tau)\,d\tau\right) T \exp\left(-\sum_k^n \int_{t_0}^t q_k(\tau)\,\hat{\mathcal{P}}_k\,d\tau - \int_{t_0}^t U(x)\,d\tau\right). \quad (56)$$

Following notations are used:

$$\mathcal{D}q(\tau) = \prod_{k=1}^n \mathcal{D}q_k(\tau), \quad \mathcal{D}q_k(\tau) = \prod_{\tau=t_0}^t \sqrt{\frac{d\tau}{2\pi}}\,dq_k(\tau), \quad (57)$$

symbol 'T' specifies chronological ordering of operators.

Until now the layout repeats those given in [12]. For further transformations of operator exponent in (56) we will use methods given in Appendix B. Hence operator in T-exponent (56) we will write as a sum of two terms:

$$\hat{h}_0(\tau) = \sum_{i=1}^n \left(\sum_{k=1}^n B_{ik}(x) q_k(\tau)\right) \frac{\partial}{\partial x_i},$$

$$\hat{h}_1(\tau) = -\sum_{k=1}^n q_k(\tau) p_k(x) + U(x).$$

Effect of the operator in (56) we will represent according to the formula (72) given in Appendix B. It is obvious that one needs to determine the effect of operators $\Theta(\tau)$ and $\Theta(\tau)^{-1}$ on variable x_i:

$$x_i(\tau) = \Theta(\tau)^{-1} x_i \Theta(\tau), \quad i \in \{1,\ldots,n\}, \quad \tau \in [t_0, t]. \quad (58)$$

By differentiating left and right parts (58) with respect to τ we obtain the following system of differential equations for value $x_i(\tau)$:

$$\dot{x}_i(\tau) = \Theta(\tau)^{-1}\big(\hat{h}_0(\tau)\,x_i - x_i\,\hat{h}_0(\tau)\big)\Theta(\tau) = \sum_{k=1}^n B_{ik}(x(\tau))\,q_k(\tau), \quad i \in \{1\ldots,n\}. \quad (59)$$

Here the properties of (73) are used (Appendix B).

The system of equations (59) we will also write in the integral form:

$$x_i(\tau) = x_i - \sum_{k=1}^n \int_\tau^t B_{ik}(x(\tau))\,q_k(\tau)\,d\tau, \quad i \in \{1,\ldots,n\},$$

where it is taken into account that $x_i(t) = x_i$, $i \in \{1,\ldots,n\}$.

This way effect of differential operators in (56) is applied and for (51) we obtain

$$K(x, x_0, t, t_0) = \int \mathcal{D}q(\tau) \exp\left(-\frac{1}{2}\sum_{k=1}^n \int_{t_0}^t q_k^2(\tau)\,d\tau\right)$$
$$\times \exp\left(\sum_{k=1}^n \int_{t_0}^t q_k(\tau)\,p_k(x(\tau))\,d\tau - \int_{t_0}^t U(x(\tau))\,d\tau\right) \prod_i^n \delta\big(x_i(t_0) - x_{i0}\big). \quad (60)$$

Next step we perform variable substitution [25] in the path integral (60):

$$x_i(\tau) = x_i - \sum_k^n \int_\tau^t B_{ik}(x(\tau))\,q_k(\tau)\,d\tau = x_i - \int_\tau^t \nu_i(\tau_1)\,d\tau_1, \quad i \in \{1,\ldots,n\}. \quad (61)$$

Differentiating (61) with respect to τ we obtain the following connection between variables $q_i(\tau)$ and $\nu_i(\tau)$ $(i \in \{1,\ldots,n\})$:

$$\sum_{k=1}^n B_{ik}(x(\tau))\,q_k(\tau) = \nu_i(\tau), \quad i \in \{1,\ldots,n\}. \quad (62)$$

As a result, for transition probability density (51) we obtain solution in a form of path integral

$$K(x,x_0,t,t_0)=\int\mathcal{D}\nu(\tau)J\exp\biggl(-\frac{1}{2}\sum_{i,k=1}^{n}\int_{t_0}^{t}\nu_i(\tau)\,\Sigma^{-1}_{ij}(x(\tau))\,\nu_k(\tau)\,d\tau\biggr)$$
$$\times\exp\biggl(\sum_{i,k}^{n}\int_{t_0}^{t}p_k(x(\tau))\,B^{-1}_{ki}(x(\tau))\,\nu_i(\tau)\,d\tau-\int_{t_0}^{t}U(x(\tau))\,d\tau\biggr)\prod_{i=1}^{n}\delta\biggl(x_i-x_{0i}-\int_{t_0}^{t}\nu_i(\tau)d\tau\biggr). \quad (63)$$

Values $p_k(x)$, $k\in\{1,\dots,n\}$ and $U(x)$ are defined in formulas (54) and (55). Jacobian J of variable substitution (62) is defined in Appendix C. Elements of functional measure $\mathcal{D}\nu(\tau)$ are given in formula (57).

Let us perform a number of transformations in formula (63). In particular, the term in exponent (63) and the Jacobian term (75) we will transform the following way, taking into account equation for $p_k(x(\tau))$ (54)

$$\sum_{i,k=1}^{n}p_k(x(\tau))\,B^{-1}_{ki}(x(\tau))\,\nu_i(\tau)+\frac{1}{2}\sum_{k,i,j=1}^{n}\frac{\partial B_{kj}(x(\tau))}{\partial x_k}B^{-1}_{ji}(x(\tau))\nu_i(\tau)$$
$$=\sum_{i,j=1}^{n}\nu_i\,\Sigma^{-1}_{ij}(x(\tau))\biggl(A_j(x(\tau))-\frac{1}{2}\sum_{k}^{n}\frac{\partial\Sigma_{kj}(x(\tau))}{\partial x_k}\biggr), \quad (64)$$

where matrix $\Sigma(x)$ is defined in formula (3). For $U(x)$ (55) after similar transformations we obtain

$$U(x(\tau))=\frac{1}{2}\sum_{i,j=1}^{n}A^c_i(x(\tau))\,\Sigma^{-1}_{ij}(x(\tau))\,A^c_j(x(\tau))+\frac{1}{2}\sum_{i=1}^{n}\frac{\partial A^c_i(x(\tau))}{\partial x_i}$$
$$+\frac{1}{8}\sum_{i,j,k=1}^{n}\frac{\partial B_{jk}(x(\tau))}{\partial x_j}\frac{\partial B_{ik}(x(\tau))}{\partial x_i}+\frac{1}{4}\sum_{i,j,k=1}^{n}B_{ik}(x(\tau))\frac{\partial^2 B_{jk}(x(\tau))}{\partial x_i\partial x_j}. \quad (65)$$

The following notations were used

$$A^c_i(x(\tau))=A_i(x(\tau))-\frac{1}{2}\sum_{j=1}^{n}\frac{\partial\Sigma_{ij}(x(\tau))}{\partial x_j}. \quad (66)$$

The last three terms in (65) can also be reduced to the form

$$\frac{1}{2}\sum_{i=1}^{n}\frac{\partial A_i(x(\tau))}{\partial x_i}-\frac{1}{8}\sum_{i,j=1}^{n}\frac{\partial^2\Sigma_{ij}(x(\tau))}{\partial x_i\partial x_j}-\frac{1}{8}\sum_{i,j,k=1}^{n}\frac{\partial B_{jk}(x(\tau))}{\partial x_i}\frac{\partial B_{ik}(x(\tau))}{\partial x_j}.$$

Term (64) and the first terms in (65) we join with the term $\sim\nu_i(\tau)\nu_j(\tau)$ in exponent (63). As the result of mentioned transformations we obtain the path integral (4).

B. Appendix

Let us consider operator T-ordered exponent

$$R=T\exp\biggl(-\int_{t_0}^{t}\hat{h}(\tau)\,d\tau\biggr). \quad (67)$$

As it is known [7], formula (67) is a form of the following Dyson series

$$R=1-\int_{t_0}^{\tau}\hat{h}(\tau_1)\,d\tau_1+\int_{t_0}^{\tau}\int_{t_0}^{\tau_1}\hat{h}(\tau_1)\,\hat{h}(\tau_2)\,d\tau_1\,d\tau_2+\dots. \quad (68)$$

Operators $\hat{h}(\tau)$ at different moments of time do not commutate

$$\hat{h}(\tau_1)\,\hat{h}(\tau_2)-\hat{h}(\tau_2)\,\hat{h}(\tau_1)\neq 0.$$

Effect of operator exponent (67) implies descending ordering of the operators $\hat{h}(\tau)$ by time variable $\tau_1>\tau_2>\dots$ (68).

Let us set operator $\hat{h}(\tau)$ as a sum of two terms

$$\hat{h}(\tau)=\hat{h}_0(\tau)+\hat{h}_1(\tau).$$

The operator exponent (67) we will write in the form

$$R=U(t)\,T\exp\biggl(-\int_{t_0}^{t}\hat{h}'_1(\tau)\,d\tau\biggr). \quad (69)$$

Here the following operator notations were used:

$$U(\tau) = T\exp\left(-\int_{t_0}^{\tau} \hat{h}_0(\tau_1)d\tau_1\right), \quad \hat{h}_1'(\tau) = U(\tau)^{-1}\hat{h}_1(\tau)\,U(\tau),$$

where $U(\tau)^{-1}$ denotes inverse operator. Operator $U(\tau)$ satisfies differential equation:

$$\dot{U}(\tau) = -\hat{h}_0(\tau)\,U(\tau), \quad U(t_0) = I,$$

where I is a unary operator. For the inverse operator $U^{-1}(\tau)$ we have the following equation

$$\frac{dU(\tau)^{-1}}{d\tau} = U(\tau)^{-1}\hat{h}_0(\tau), \quad U(t_0)^{-1} = I.$$

Operator exponent (69) we also write in the form

$$R = T\exp\left(-\int_{t_0}^{t} \hat{\hat{h}}_1(\tau)\,d\tau\right)U(t), \tag{70}$$

where

$$\hat{\hat{h}}_1(\tau) = \Theta(\tau)^{-1}h_1(\tau)\,\Theta(\tau), \quad \Theta(\tau) = U(\tau)\,U(t)^{-1}. \tag{71}$$

From (71) it follows that the equality $U(t) = \Theta(t_0)^{-1}$ is valid. Operator R in (70) we will write in the form

$$R = T\exp\left(-\int_{t_0}^{t} \hat{\hat{h}}_1(\tau)\,d\tau\right)\Theta(t_0)^{-1}. \tag{72}$$

For operators $\Theta(\tau)$ and $\Theta(\tau)^{-1}$ we obtain equation:

$$\begin{gathered}\dot{\Theta}(\tau) = -\hat{h}_0(\tau)\,\Theta(\tau), \quad \Theta(t) = I,\\ \frac{d\Theta(\tau)^{-1}}{d\tau} = \Theta(\tau)^{-1}\,\hat{h}_0(\tau), \quad \Theta(t)^{-1} = I.\end{gathered} \tag{73}$$

Solving equation (73) for $\Theta(\tau)$ using iteration method we obtain for operator $\Theta(\tau)$ a Dyson series (68)

$$\Theta(\tau) = 1 + \int_{\tau}^{t}\hat{h}_0(\tau_1)\,d\tau_1 + \int_{\tau}^{t}\int_{\tau_1}^{t}\hat{h}_0(\tau_1)\,\hat{h}_0(\tau_2)\,d\tau_1\,d\tau_2 + \ldots,$$

which, using ordering operator we will write in the following form

$$\Theta(\tau) = \acute{T}\exp\left(\int_{\tau}^{t}\hat{h}_0(\tau_1)\,d\tau_1\right). \tag{74}$$

Here $\acute{T}$ shows that operators $\hat{h}_0(\tau)$ in (74) are ordered ascedingly by time variable contrary to (68).

C. Appendix

Jacobian of variable substitution (62) is equal to the functional determinant

$$J = \left\|\frac{\delta q_k(\tau)}{\delta \nu_i(\tau')}\right\|.$$

Taking into account expression (62) we obtain

$$J = \left\|B_{ki}^{-1}(x(\tau))\,\delta(\tau-\tau') - \sum_{j=1}^{n}\frac{\partial B_{kj}^{-1}(x(\tau))}{\partial x_i}\,\theta(\tau'-\tau)\,\nu_j(\tau)\right\|.$$

After some transformation we write Jacobian in the following form

$$J = \left(\prod_{\tau=t_0}^{t}\frac{1}{\det(B(x(\tau)))}\right)\left\|\delta_{kk'}\,\delta(\tau-\tau') + \theta(\tau'-\tau)\sum_{i,j=1}^{n}\frac{\partial B_{k'j}(x(\tau))}{\partial x_k}B_{ji}^{-1}(\tilde{x}(\tau))\,\nu_i(\tau)\right\|.$$

Methods for calculating Jacobian are given [11, 12, 25] and as a result we obtain the following

$$J = \left(\prod_{\tau=t_0}^{t}\frac{1}{\det(B(x(\tau)))}\right)\exp\left(\frac{1}{2}\sum_{k,i,j=1}^{n}\int_{t_0}^{t}\frac{\partial B_{kj}(x(\tau))}{\partial x_k}B_{ji}^{-1}(x(\tau))\,\nu_i(\tau)\,d\tau\right). \tag{75}$$

D. Appendix

Let us consider a Fourier integral expansion of δ-function

$$\delta\big(Se^{x}-S_0\big)=\frac{1}{2\pi}\int_{-\infty}^{\infty}f(k)\,e^{ikx}dk.$$

Using inverse transform we find $f(k)$

$$f(k)=\int_{-\infty}^{\infty}\delta\big(Se^{x}-S_0\big)\,e^{-ikx}dx=\frac{1}{S_0}e^{ik\ln(S/S_0)}.$$

As a result we obtain

$$\delta\big(Se^{x}-S_0\big)=\frac{1}{2\pi S_0}\int_{-\infty}^{\infty}e^{ik(x+\ln(S/S_0))}dk.$$

E. Appendix

The path integral in (33) form is known for the problem of radial oscillator and has analytic solution [7, 12, 28]. In "velocity" variables for the integral we find:

$$\int\mathcal{D}q(\tau)\exp\left(-\frac{1}{2}\int_{t_0}^{t}\dot z(\tau)^2d\tau\right)\exp\left\{-\frac{1}{2}\omega^2\int_{t_0}^{t}z(\tau)^2d\tau-\frac{1}{2}\left(\lambda^2-\frac{1}{4}\right)\int_{t_0}^{t}\frac{1}{z(\tau)^2}d\tau\right\}\delta\big(z(t_0)^2-z_0^2\big)$$
$$=\frac{1}{2}\exp\left\{-\frac{1}{2}(z^2+z_0^2)\omega\coth(\omega(t-t_0))\right\}\sqrt{\frac{z}{z_0}}\frac{\omega}{\sinh(\omega(t-t_0))}\,I_\lambda\left(\frac{zz_0\,\omega}{\sinh(\omega(t-t_0))}\right),$$
$$z(\tau)=z-\int_{\tau}^{t}q(\tau_1)\,d\tau_1,$$

where $I_\lambda(x)$ is the modified Bessel function.

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Розв'язок розширених моделей Гестона в методі функціонального інтегрування

Янішевський В. С.

Національний університет "Львівська політехніка", вул. С. Бандери, 12, 79013, Львів, Україна

Метод функціонального інтегралу розвинутий в роботі автора застосовано до розв'язання розширених моделей Гестона. Знайдено густини умовних ймовірностей для моделей Гестона зі стохастичними відсотковими ставками, що задаються стохастичними процесами Васічека і CIR. Для обох випадків знайдені формули ціни опціонів. Знайдені точні розв'язки мають місце за відсутності кореляції вінерівського процесу відсоткової ставки з вінерівськими процесами ціни активу чи волатильності. У випадку наявності вказаної кореляції моделі не мають точного розв'язку, що узгоджується з результатами інших робіт.

Ключові слова: *стохастична модель; умовна ймовірність; функціональний інтеграл; модель Гестона; розширена модель Гестона.*