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Exploring chaotic dynamics with absolute-embedded sinusoidal nonlinearity in a sinusoidal-enhanced Van der Pol oscillator

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The present study aims to analyze the chaotic behavior of a simple electrical circuit with a nonlinear resistor and an absolute value in its sinusoidal nonlinearity function. This type of circuit is considered fundamental, because it contains both nonlinear capacitance and resistance. This study investigates into various aspects of the circuit's behavior, with a particular focus on its chaotic properties such as bifurcation, periodicity, resonance, and Lyapunov exponent analysis. It is essential to highlight that, in addition to its chaotic behavior, the system also exhibits a stable equilibrium point and a chaotic attractor.

Keywords: *Van der Pol Oscillator; chaos; bifurcation; periodic; resonance.*

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1. Introduction

Researchers studied complex, unpredictable dynamic systems characterized by intricate details and sensitivity to initial conditions, as evidenced by their exploration of similar features in dynamic systems [1]. A remarkable characteristic of such systems is their chaotic nature. Chaos refers to a phenomenon in dynamical systems theory while a system exhibits chaotic behavior for a limited time before stabilizing into a periodic or quasi-periodic state. This concept has been extensively studied in various fields including physics, mathematics, engineering, and biology.

In 1983, Celso Grebogi, Edward Ott, and James Yorke published a pioneering paper on chaos that demonstrated the finite lifetime of chaotic attractors in the Lorenz system [2, 3]. Despite the inherent unpredictability of the Lorenz system, the authors demonstrated that chaotic behavior can be predicted by analyzing the dynamics of unstable periodic orbits. This groundbreaking work introduced the concept of unstable periodic orbits and highlighted their significance in the emergence of chaos in dynamic systems.

Grebogi, Ott, and Yorke have inspired researchers to continue investigating transitory chaos in different systems, including mechanical, electronic, and biological systems. This field has made significant progress in using transient chaos to enhance a system's sensitivity to small changes, which is applied in various applications such as cryptography and secure communications.

Researchers have recently begun exploring the relationship between chaos and synchronization in complex networks. This study found that chaos can help promote synchronization in networks of coupled oscillators, which has potential applications in fields such as power systems and communication networks.

In the field of dynamical systems theory, extensive research has been conducted on chaos. The concept of transitory chaos has advanced our understanding of chaotic systems, including the Ruelle–Takens scenario and unstable periodic orbits. Additionally, the characteristics of transient chaos, such as network synchronization and cryptography, have been applied to various real-world situations.

This study examines a straightforward circuit that features brief chaotic oscillations using sine-wave nonlinearity. Research on the design, implementation, and analysis of oscillator qualities has gained considerable attention in recent years. Oscillators with a propensity for chaos have attracted interest owing to their potential applications. An oscillator is an electronic device that produces alternating recurring waves without requiring any input. In this process, power signals are transformed into current signals, which is a fundamental characteristic frequently utilized in electrical devices [4–9].

A chaotic oscillator is a mathematical representation of a system that exhibits high sensitivity to the initial conditions and evolves in an unpredictable and seemingly random manner. This is in contrast to an optimization algorithm [10], that is a computational strategy designed to iteratively refine solutions to specific problems [11] with the aim of discovering the optimal set of parameters or configurations [12]. Numerous natural and artificial systems, such as weather patterns, electrical circuits, and biological processes, exhibit chaotic behavior. Chaotic oscillators have applications in various disciplines, including cryptography, in which they can generate reliable encryption keys. Additionally, they are employed in physics to analyze the behavior of complex systems and in biology to model the dynamic behaviors of populations of organisms.

Various control strategies, including feedback, time-delay feedback, and backstepping mechanisms [13–15], have emerged as crucial tools for managing chaotic behaviors, such as system perturbations and noise [16–18]. Stabilization has been successful in controlling chaos through either the most likely technique [19–23] or sampled method [24]. The effectiveness of these methods is evident in the shifts in the positions of the system points in the phase view and alterations in the attractor of the system [25–27].

This paper introduces a novel chaotic circuit design that integrates a Van der Pol oscillator and sine-wave nonlinearity. The proposed system, inspired by the existing chaotic circuit literature, generates unpredictable oscillations with a straightforward implementation. The goals of this study are as follows: to develop an easy-to-use non-autonomous circuit that exhibits chaos and its qualitative characteristics, such as bifurcation and Lyapunov exponents. To achieve these goals, a mathematical model of the proposed oscillator is presented and discussed in detail in Section 2. Section 3 explores and offers conclusions for future research.

2. Circuit realization and chaos

A capacitor (C), inductor (L), nonlinear resistance (R), external periodic forcing, and a nonlinear element, notably the sinusoidal non-linearity, make up the self-maintaining electrical circuit. As shown in Figure 1, they were all interconnected in series. This is a variation of the Van der Pol oscillator.

When Kirchoff's law is applied to this circuit, the voltage v across capacitor C and the current i via inductor i_L are governed by two first-order non-autonomous differential equations,

$$\begin{aligned} C\frac{dv_C}{dt} &= i_L + f\sin(\Omega t), \\ L\frac{di_L}{dt} &= -v_C - Ri_L\big(v_C^2 - g(v_C)\big), \end{aligned} \tag{1}$$

where f denotes the magnitude of an external periodic force, and is Ω its angular frequency. The Chua's diode's characteristic curve equation is

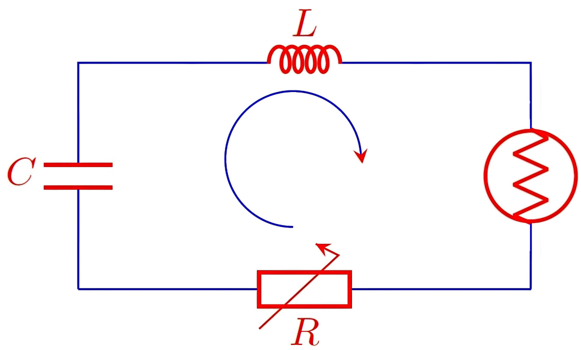

Fig. 1. Circuit.

$$g(v_C) = \frac{2a}{\pi}\sin^{-1}\left(\sin\frac{2\pi|v_C|}{p}\right).$$

By rescaling Eq. (1) as $v_C = xB_p$, $i_L = GyB_p$, $G = \frac{1}{R}$, $\omega = \frac{\Omega C}{G}$ and then, and by redefining τ as t, the following set of normalized equations is obtained:

$$\begin{aligned} \dot{x} &= cy + F\sin(\omega t), \\ \dot{y} &= -x - \mu\left(x^2 - g(x)\right)y. \end{aligned} \tag{2}$$

Equations (2) present the dynamics, which are determined by the parameters a, b, c, μ, w, and F.

Lyapunov exponents are the numbers used to compute the temporal separation between two adjacent orbits in the phase space under favorable conditions [28]. An n-dimensional system contains n Lyapunov exponents. The presence of at least one positive Lyapunov exponent in the system indicated chaos. If the maximum Lyapunov exponent is negative, the orbits will converge in time, and the system will be insensitive to these conditions. If it is positive, the system has a sensitive initial condition dependency, and the distance between adjacent orbits increases exponentially; thus, it is chaotic. The Lyapunov exponent of a dynamical system can detect chaos and assess the stability of the system.

In this study, the Wolf algorithm [28] was employed to calculate the Lyapunov exponents under specific parameter values: $a = 1.25$, $c = 3.5$, $\mu = 2/3$, $w = 1$, and $F = 1.5$, with observation periods $(T) = 100$, and a sampling time $\Delta t = 0.5$. The analysis revealed the existence of two positive Lyapunov exponents, which suggests that small changes in the initial conditions can result in significantly different outcomes over time, a characteristic indicator of chaos. The positive values of the Lyapunov exponents indicate exponential divergence of nearby trajectories, further confirming the chaotic nature of the system.

Figure 2 depicts the Lyapunov exponent for the system (1) with factors $F = 1.25$ and initial conditions $(x, y) = (0.1, 0.1)$ and $(x, y) = (0.1, 0.5)$.

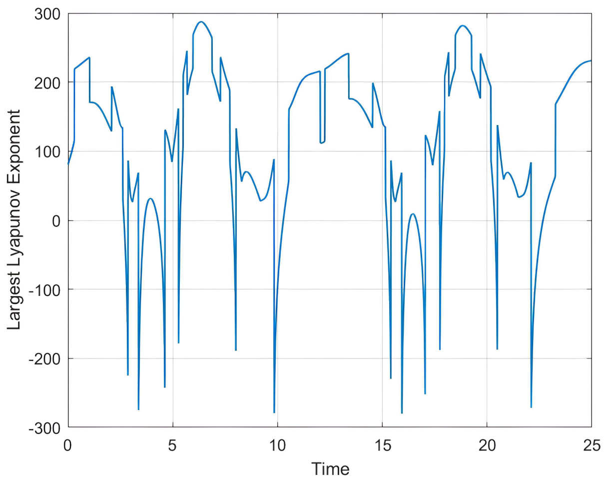

Fig. 2. Largest Lyapunov exponent behavior over time.

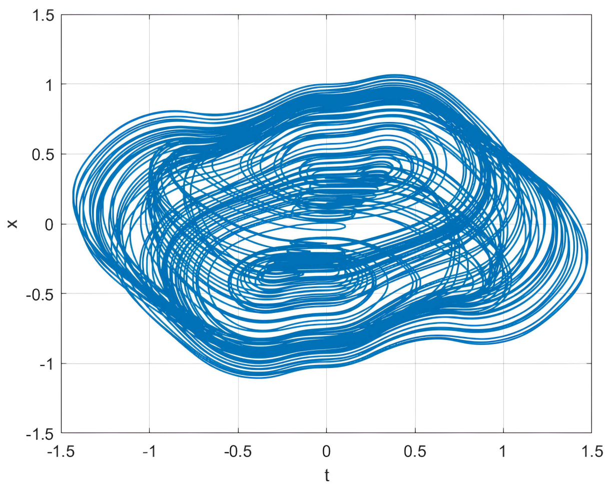

Fig. 3. Chaotic oscillator circuit diagram.

The use of a two-dimensional projection to display the phase space is commonly referred to as a phase portrait, which illustrates the instantaneous state of each state variable. Each point in the phase portrait represents a fixed-point solution, whereas closed and distinct curves represent periodic and chaotic solutions, respectively.

Figure 3 presents a detailed examination of the inherent chaos present in the proposed oscillator, showing the behavior of the system under specific parameter values, namely $a = 1.25$, $c = 3.5$, $\mu = 2/3$, $\omega = 1$, $F = 1.5$, and $w = 1.0$.

The new chaotic system with line equilibrium, represented by equation (1), demonstrates increased complexity owing to its chaotic behavior. This system can be examined using various techniques, such as Lyapunov exponents, phase portraits, power spectra, time series, and Poincare maps, in both local and global settings to investigate its periodic and chaotic behavior. It was found that, for certain parameter values, the circuit exhibited only one long-term motion, whereas for others, it exhibited multiple motions. The term "bifurcation" refers to the phenomenon in differential equations when the number of solutions changes based on parameter variation [29]. If the number of solutions to differential equations changes because of the parameter values F and c, then according to suitable the first-order differential equations (2), the nature of the solutions of the circuit (1) may abruptly change, resulting in a phenomenon known as "bifurcations" [29]. Typically, one type of motion reduces stability at a critical parameter value as it differs gradually, resulting in a new type of stable motion. Parameter values that occur at bifurcations are referred to as "bifurcation points" or "bifurcation values" [29]. This causes circuit (1) to become chaotic.

Detecting bifurcations in circuit (1) can be easily accomplished by examining the variables F and c. Specifically, when parameter c lies between 2.5 and 3, circuit (1) exhibits a bifurcation nature, as illustrated in Figure 4, with the conditions $(x, y) = (0.1, 0.1)$. Similarly, when parameter c lies between 3 and 4, circuit (1) also exhibits a bifurcation nature, as illustrated in Figure 5 with the conditions $(x, y) = (0.1, 0.1)$. In Figure 6 shows bifurcation for c lies between 4 and 5. Moreover, when the parameter F lies between 1 and 2, circuit (1) exhibits bifurcation, as illustrated in Figure 7 with the condition $(x, y) = (0.1, 0.1)$. In Figure 8 shows bifurcation for F between 2 and 2.5. Figure 9 shows bifurcation for μ between 0 and 1.

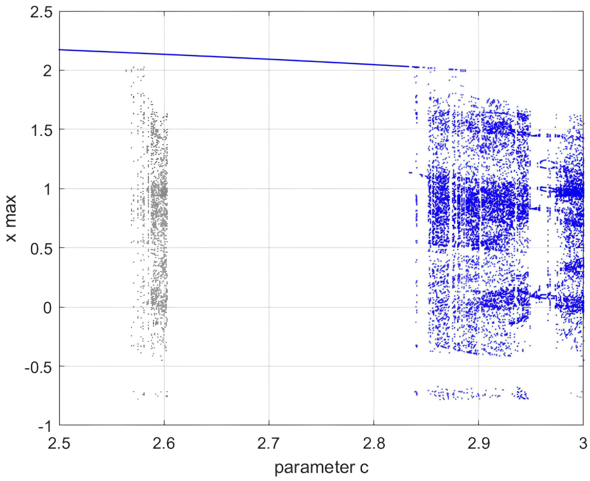

Fig. 4. Bifurcation for c in between 2.5 and 3.

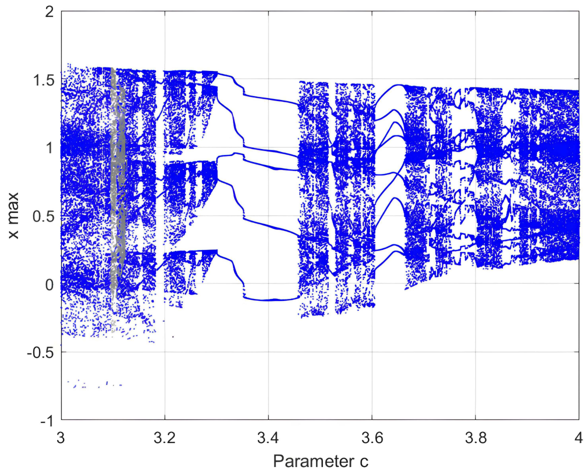

Fig. 5. Bifurcation for c in between 3 and 4.

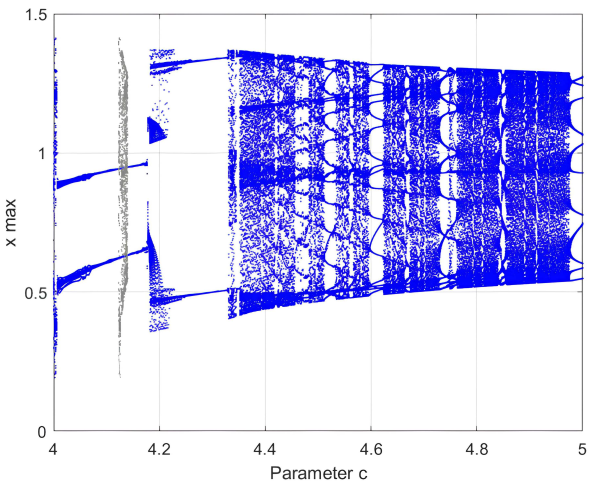

Fig. 6. Bifurcation for c in between 4 and 5.

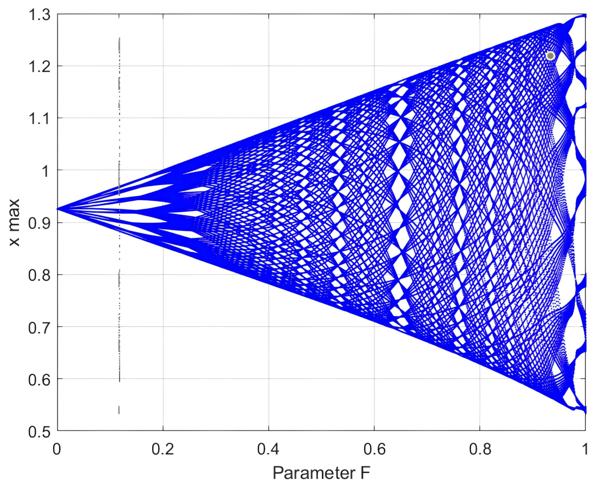

Fig. 7. Bifurcation for F in between 1 and 2.

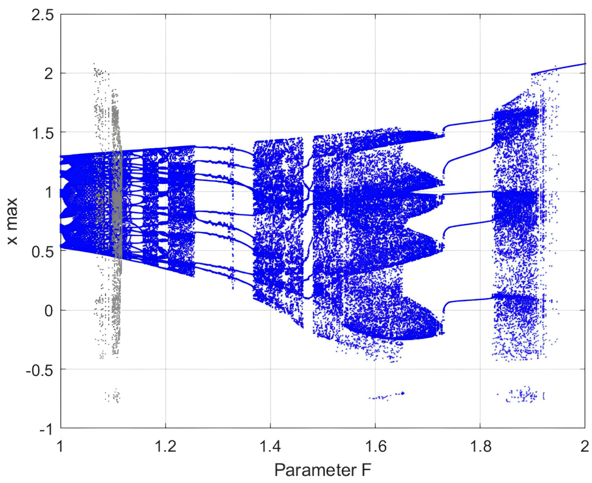

Fig. 8. Bifurcation for F in between 2 and 2.5.

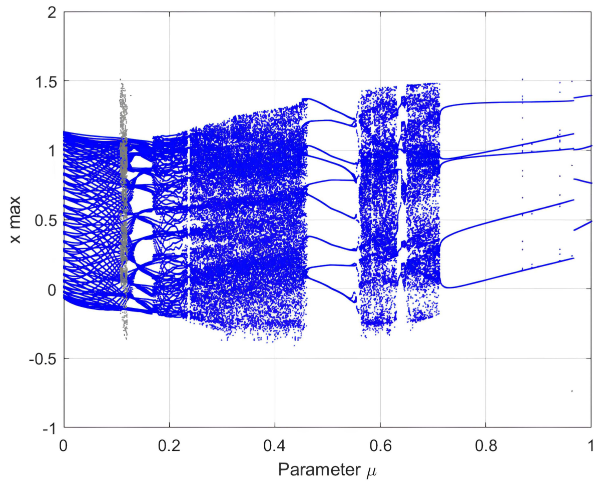

Fig. 9. Bifurcation for μ in between 0 and 1.

Figures 10 and 11 display the phase portrait of variables x and y along with the local Lyapunov exponent and contour intersections. The parameter values used for Figure 10 are $a = 1.25$, $c = 5$, $\mu = 2.3$, $\omega = 1$, $F = 1.5$, and $w = 1.0$. These values produced a striking phase portrait, as shown in Figure 10. In contrast, Figure 11 captivatingly portrays the same phase portrait, but with the same parameter values.

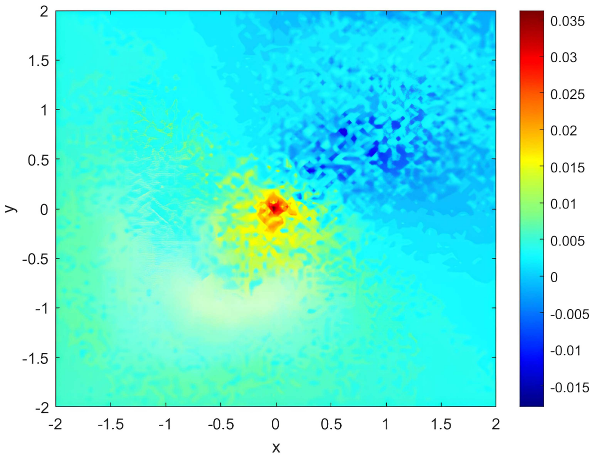

Fig. 10. Phase portrait with local Lyapunov.

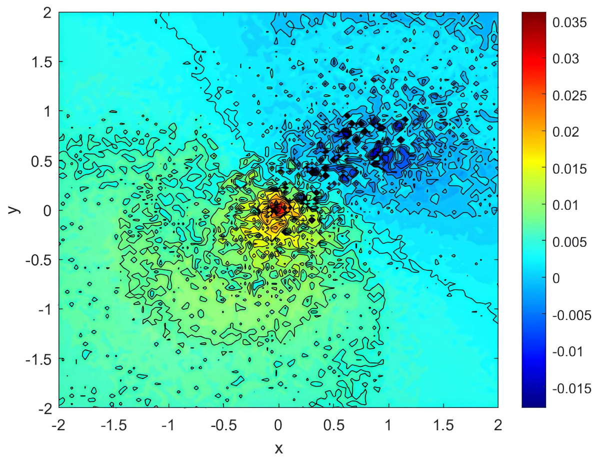

Fig. 11. Phase portrait with local Lyapunov and contour lines.

In both figures, the colors represent the different values of the local Lyapunov exponent. Blue, green, yellow, and red represent low, intermediate, moderately high, and high values of the local Lyapunov exponent, respectively. Higher values of the local Lyapunov exponent indicate greater sensitivity to the initial conditions and more chaotic behavior in the system.

In Figure 10, the concentration of red-yellow regions in the center of the phase portrait indicates that the trajectories are most tightly packed in this area, suggesting that the system is the most chaotic in this region. In contrast, the blue and green regions were found in a smaller portion of the portrait, where the trajectories were less spread, suggesting that the system was less chaotic in these regions.

Figure 11 shows the spatial distribution of the local Lyapunov exponents using a color-coded contour plot. The black contour lines enhance the visibility of the patterns, with blue and green shades representing areas with lower local Lyapunov exponent values. These regions exhibit a slower divergence of nearby trajectories, indicating a less chaotic behavior. Conversely, the yellow and red shades represent areas of higher local Lyapunov exponent values, indicating a faster divergence of nearby trajectories and more chaotic behavior.

The boundaries of distinct regions with different Lyapunov exponent values are demarcated by black contour lines, presenting a visually clear representation of the distribution of chaoticity in the phase space. The color gradient seamlessly transitions from blue (low values) to yellow, and finally to red (higher values), incorporating intermediate shades such as green to effectively communicate the variation in Lyapunov exponent values across the $x - y$ plane.

Figures 12 and 13 display the intricate dynamics of the proposed chaotic circuit, showing its remarkable ability to transition between an ordered periodicity and intricate chaos. The figures clearly illustrate the periodic characteristics of the system, highlighting the presence of discernible patterns during its complex oscillations. Additionally, it captures the system's chaotic essence, emphasizing its erratic nature and sensitivity to initial conditions, which are the defining features of chaotic dynamics.

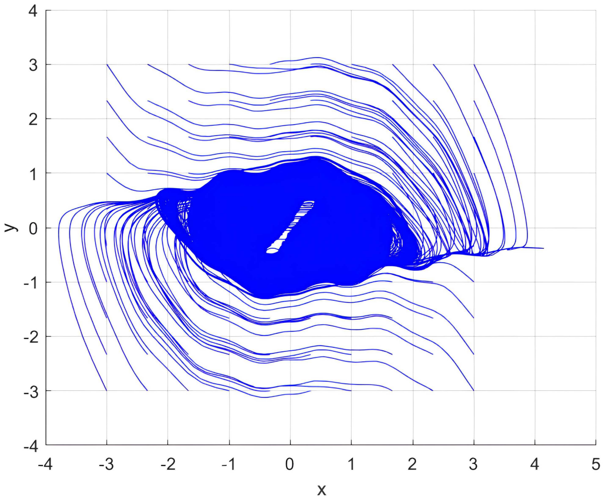

Fig. 12. Periodic nature of the chaotic system: potential periodic behavior.

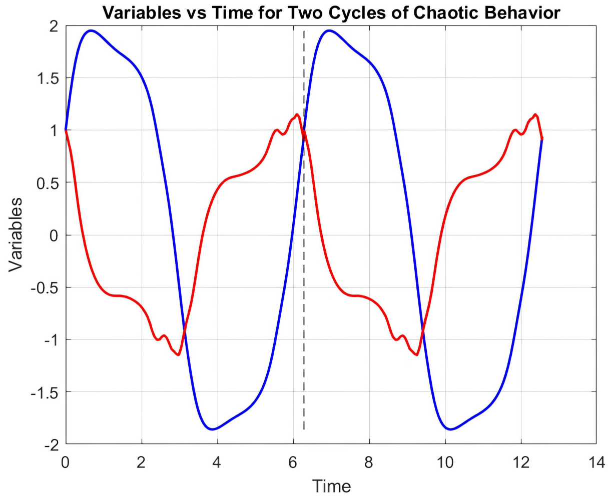

Fig. 13. Periodic nature of the chaotic system: variables vs time for two cycles of periodic behavior.

In Figure 13, the blue lines indicate the potential for periodic behavior within the proposed chaotic system. This suggests that the system may exhibit periodic trends under specific initial conditions.

Figure 14 depicts the resonant behavior of the proposed chaotic circuit, which emerges from the interplay between the nonlinearities of the chaotic system. This figure illustrates how the system reacts to external stimuli by amplifying the response at specific frequencies, emphasizing its potential use in frequency filtering and signal processing. The intricate relationship between the ordered periodicity, chaotic unpredictability, and resonant behavior highlights the circuit's adaptability and potential applications in various sectors, including secure communication systems and chaos-based cryptography.

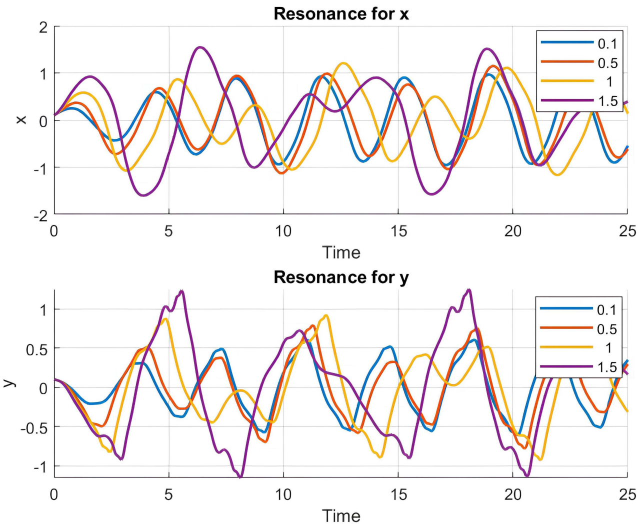

Fig. 14. Chaotic oscillator resonance diagram.

3. Conclusion

This research involves constructing a circuit comprising a capacitor, resistor, inductor, and absolute sin-wave nonlinearity. In addition to existing models for the inductor, capacitor, and resistor, this study employs Kirchhoff's voltage-current law to derive the dynamic equation for the system. This study investigated the dynamic properties of a novel chaotic oscillator, demonstrating its chaotic nature. Bifurcation diagrams and Lyapunov exponents were employed to discuss the oscillator, and classical dynamics were used to examine the stability of the equilibrium point. The analysis of Lyapunov exponents, complexity, and bifurcation diagrams reveals that the chaotic system exhibits intricate dynamical behavior as the initial conditions and parameters change.

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Дослідження хаотичної динаміки з абсолютно-вбудованою синусоїдальною нелінійністю в синусоїдально-покращеному осциляторі Ван-дер-Поля

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Метою цього дослідження є аналіз хаотичної поведінки простого електричного кола з нелінійним резистором і абсолютним значенням його синусоїдальної нелінійної функції. Цей тип схеми вважається основним, оскільки містить як нелінійну ємність, так і опір. Це дослідження досліджує різні аспекти поведінки схеми, з особливим акцентом на її хаотичних властивостях, таких як біфуркація, періодичність, резонанс і аналіз показника Ляпунова. Важливо підкреслити, що, крім хаотичної поведінки, система також демонструє стабільну точку рівноваги та хаотичний атрактор.

Ключові слова: *осцилятор Ван дер Поля; хаос; біфуркація; періодичний; резонанс.*