MATHEMATICAL MODELING AND COMPUTING, Vol. 12, No. 1, pp. 212–220 (2025)

Modeling short term interest rates using the Vasicek and Cox–Ingersoll–Ross models

Md Sa'at A. S.[1], Shair S. N.[1,2], Md Lazam N.[1,2], Yusof A. Y.[1], Mohd Amin M. N.[1,2], Ibrahim R. I.[3], Mohd Ghani N. A.[1]

[1] *School of Mathematical Sciences, College of Computing, Informatics and Mathematics, Universiti Teknologi MARA (UiTM), 40450, Shah Alam, Selangor*
[2] *Research Interest Group of Actuarial Risk, Analytics and Takaful, Universiti Teknologi MARA (UiTM), 40450, Shah Alam, Selangor*
[3] *Faculty of Science and Technology, Universiti Sains Islam Malaysia, Bandar Baru Nilai, 71800 Nilai, Negeri Sembilan, Malaysia*

(Received 19 November 2024; Revised 17 February 2025; Accepted 19 February 2025)

In modeling future uncertainties, the time value of money effect is often minimally addressed. Many models assume constant rates, leading to potential errors in financial instrument pricing. This study explores continuous-time interest rate models to capture future uncertainties of interest rates. Two stochastic interest rate models, the Vasicek and Cox–Ingersoll–Ross (CIR), will be adopted, and their forecast performance will be evaluated. Using the Maximum Likelihood Estimation (MLE), the models are fitted to Kuala Lumpur Interbank Offered Rate (KLIBOR) data (1-month, 3-month, and 12-month rates). Subsequently, the goodness-of-fit and forecast accuracies of both models were analysed. Results show that the Vasicek model is superior based on AIC, BIC, MSE, RMSE, and MAPE measures. The Vasicek model outperforms CIR overall, especially for 12-month rates. Finally, this study estimates zero-coupon bond prices and develops the term structure of interest rates, revealing an inverted yield curve.

Keywords: *stochastic modeling; interest rate risk; Vasicek model; Cox–Ingersoll–Ross model.*

2010 MSC: 91G30, 60H10 **DOI:** 10.23939/mmc2025.01.212

1. Introduction

Modeling interest rate risk is crucial for accurate actuarial valuations of financial instruments. In recent years, there have been extensions of interest rate models, which include shifting from a deterministic to a non-deterministic approach. The shift is deemed necessary since the deterministic approach fails to capture future uncertainties and changes, that could result in errors in pricing and reserve estimation of financial products. Some actuarial valuations involve insurance products that are offered for a long period of time, such as whole life insurance or an annuity product. Thus, failing to account for fluctuations in interest rates in the long term could also cause financial problems for the company. According to [1], any unexpected change in interest rate movements could lead to asset-liability mismatching that will affect the available funds or surplus in a portfolio.

There are a few methods that could account for long-term interest rate changes. Traditionally, interest rate risk is measured using deterministic approaches. For instance, interest rates were hedged via immunization or optimal matching of the assets and liabilities of a company, an approach proposed by [2] through the Redington's theory of immunization. References [3–5] extended this method further to include a non-flat yield curve and immunization for multiple liabilities. These deterministic models however fail to account for future changes or uncertainties, thereby constraining the relevance and robustness of the future measures and limiting its overall effectiveness in dynamic financial environments.

This work was supported by the grant FRGS/1/2023/STG06/UITM/02/6.

Modeling interest rates using a stochastic approach has the advantage to capture the changes in interest rates that evolve through time. Applying such models is important, especially in long-term actuarial valuations as government bonds. Due to high security and their nature of having a longer maturity time, interest rates are commonly used as discount factors when it comes to valuing government bonds. One of the earliest stochastic models that consider the movement in interest rates is the continuous-time stochastic process model known as the generalized Wiener process or Brownian motion, as explained in [6]. The model consists of two fixed parameters that represent drift and volatility. The Brownian motion is extended to geometric Brownian motion (GBM) assuming that a variable follows the lognormal distribution [6]. More commonly, the extension is known as an Itô process, where two parameters are set to be functions dependent on the value of an underlying variable and time [7]. This model is extensively used in modeling stock prices and indices. ItB's assumed that continuously compounded rates of return are normally distributed, which in turn makes stock prices to be lognormally distributed. This assumption works well for stock prices as the lognormality assumption means that stock prices will not have negative values. However, the process has some limitations, because the normality assumption for rates of return means that interest rates can be negative, which is not ideal.

The Itô process is extended to three one-factor equilibrium models where economic variables are used to form a process for short rates [8]. This means that one factor considered is that the price of a zero-coupon bond price is determined by short-term interest rates only over the maturities [9,10]. The equilibrium models include the Rendleman–Bartter, the Vasicek and the Cox–Ingersoll–Ross (CIR). The latter model is the most sophisticated as it can eliminate four main concerns when it comes to using continuous-time stochastic model: (a) the drift parameter might not be applicable for use in longer periods, (b) the result of the model could lead to negative force of interest, (c) constant volatility is assumed for large or small force of interest and (d) where the force of interest does not possess mean reversion property. The Rendleman–Bartter model succeeds in eliminating only (b) and (c) and the Vasicek model can eliminate only (a) and (d). Most stochastic interest rate models are specific instances of the affine term structure model (ATSM) and can provide analytical approaches for pricing zero-coupon bonds [11]. The Vasicek model for example, gives protection against interest rate movement. Although the Vasicek model assumes constant volatility for the force of interest rate and allows for a negative force of interest, it is important to highlight its mean reversion property and tractability. The CIR model is limitedly used in previous literature, probably because it is more complicated. The Rendleman–Bartter model assumes interest rates follow geometric Brownian motion which is more suitable for stock prices rather than interest rates. This is less ideal as stock prices do not behave like interest rates, where rates seem to revert to a long-term average level over time.

The application of stochastic interest rate models has been growing over the years. For example, Reference [12] apply the Vasicek model to the European Interbank Offered Rate (EURIBOR), Tallinn Interbank Offered rate (TALIBOR), and Canadian zero-coupon bond yields to estimate short rates and observe that the accuracy of the model is high. Reference [13] utilize three short rate models, specifically, the Vasicek, the CIR, and the 3/2 model to the one-year US deposit rates. The paper concludes that, although all three models are able to fit into the data, the Vasicek model is the least complex and easily fitted. Reference [14] apply multiple short rate models, including the Vasicek and CIR, to the Vietnamese Treasury bill rates and conclude that other models are superior to both the Vasicek and CIR. Reference [1] apply the CIR model to the Australian zero-coupon bond yield rates with maturities ranging from three to 120 months using the General Method of Moments approach to estimate the prices of longevity bonds for the purpose of immunizing and hedging post-retirement income annuities. The implementation of the Vasicek model can also be referred to in the research by [11] where the Australian zero-coupon bond discount factors with maturities spanning from three months to 30 years are used to estimate short rates for value-based longevity index estimation. Additionally, [15] apply both the Vasicek and CIR models to the Indonesian monthly rates to compare the Projected Unit Credit and the Entry Age Normal approaches for pension fund valuation. The paper concludes that entry age normal using the Vasicek model gives a lower normal cost value but the projected unit credit with the CIR model generates more profit for pension funds. Reference [16] applies the Vasicek model to the one-year Shanghai Interbank Offered Rate (SHIBOR) and observes some negative interest rates, which displays a disadvantage of the model.

The adoption using Malaysian data, however, is centered on the application of the geometric Brownian motion to forecast stock prices. For example, [17] applies four different methods, including the Vasicek, to estimate the systematic risk or the beta coefficients of Malaysian securities and conclude that the Vasicek model is the best method to forecast systematic risk for the overall Malaysian market. Reference [18] compares the Vasicek model with the Autoregressive Integrated Moving Average (ARIMA) and GBM to forecast the weekly stock prices using data from Bursa Malaysia. The study observes that all three forecast methods give high accuracy but concludes that the ARIMA method is the best to forecast stock prices. One specific usage of the equilibrium model can be seen in [19] which applies the Vasicek and CIR models to 1-year Malaysian Government Securities (MGS) to estimate annuity prices using stochastic interest rates. The study concludes that the pricing model has no significant difference between the Vasicek and CIR models. However, it is found that the Vasicek model is more sensitive towards changes. As the application of the stochastic model is minimal in the Malaysian context, it is highly important to further explore and adopt these models to ensure that the risks affecting future valuations are properly measured.

This paper aims to extend the applications of two commonly used stochastic models, the Vasicek and CIR using the Kuala Lumpur Interbank Offered Rate (KLIBOR) data. Three KLIBOR datasets consist of 1-month, 6-month, and 12-month interest rates are selected. The calibrated parameters for each data set are used to simulate interest rate paths. The model that gives the highest out-sample forecast accuracy is then identified. Using the projected short rates, the study then estimates the prices of zero-coupon bonds across maturities of $1, 2, 5, 10, 20$, and 30 years. A term structure of interest rates, often referred to as a yield curve, is then formed using the best-fitting model to further analyse the impact of the interest rates estimation. The subsequent sections of this article are structured as follows. Section 2 outlines the data collection process, the short rate models, the log-likelihood functions, the measures of errors and out-sample forecast accuracies, and the estimation of zero-coupon bond prices and their yields. Section 3 discusses detailed findings, parameter fittings, and further calculations. Finally, Section 4 concludes the research with some suggestions for future work.

2. Methodology

2.1. Data collection

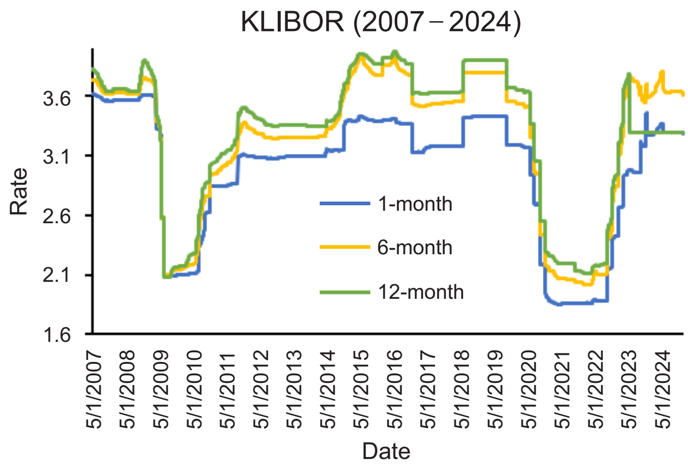

Fig. 1. The daily historical KLIBOR trends for the period of 5^{th} January 2007 to 12^{th} August 2024.

This study collects KLIBOR daily historical time series data focusing on the 1-month, 6-month, and 12-month short rates, considering the shortest, longest, and the average duration in between. The interest rates from 5^{th} January 2007 to 12^{th} August 2024 were extracted from the Financial Markets Investor Portal (FMIP) [20]. This data is reliable and of high quality, with any missing entries filled using linear interpolation. There are $4,537$ interest rates recorded for each period resulting totally of 13 611 daily historical interest rates. From Figure 1 below, it is observed that interest rates are at their lowest circa 2009 to 2010 (the Great Recession) and 2020 to 2023 (the COVID-19 pandemic). However, interest rates have gradually increased during post-recession and post-pandemic periods and showed a stable fluctuation from 2011 to 2020 and from 2023 until the present time.

The collected datasets were separated into training sets (5th January 2007 to 23rd March 2020) and validation sets (24th March 2020 to 12th August 2024) with a ratio of 70:30 for each of the time periods, respectively. The training sets calibrate the model parameters of Vasicek and CIR, and the validation sets are used to compare the actual and estimated interest rates. Once the data has been finalized, the analysis will be performed using the methodologies below and *Microsoft Excel* for data analysis and `R programming` for modelling and projections of interest rates using both models.

2.2. Modeling the short term interest rates

2.2.1. The Vasicek model

The one-factor Vasicek model for interest rates is described by the following stochastic differential equation:

$$dr(t) = a(b - r(t))\,dt + \sigma\,dZ(t), \tag{1}$$

where $r(t)$ is the short rate at time t, a is the constant mean reversion speed, b is the constant long-term average rate, σ is the constant volatility and $Z(t)$ is a standard Brownian motion.

The mean-reversion factor can be seen where $r(t)$ is drawn towards a level b at a rate a. The stochastic integral equation for $r(t)$ gives

$$r(t) = r(s)\,e^{-a(t-s)} + b\big(1 - e^{-a(t-s)}\big) + \sigma \int_s^t e^{-a(t-u)}\,dZ(u), \quad s \leqslant t. \tag{2}$$

This result shows that the interest rates may not be positive, limiting the model using. The short rate assumes the normal distribution with mean and variance respectively as follows:

$$\mathrm{E}[r(t)|r(s)] = r(s)\,e^{-a(t-s)} + b\big(1 - e^{-a(t-s)}\big), \tag{3}$$

$$\mathrm{var}[r(t)|r(s)] = \frac{\sigma^2}{2a}\big(1 - e^{-2a(t-s)}\big). \tag{4}$$

2.2.2. The Cox–Ingersoll–Ross (CIR) model

On the other hand, one-factor Cox–Ingersoll–Ross (CIR) model for interest rates is described by the following stochastic differential equation:

$$dr(t) = a\,(b - r(t))\,dt + \sigma\sqrt{r(t)}\,dZ(t). \tag{5}$$

The model maintains its mean-reverting property as the Vasicek model but assumes that the volatility of the change in the short rate is proportional to $\sqrt{r(t)}$, indicatiing that as the short rate increases, the standard deviation will increase. This property ensures that interest rates will not be negative. The stochastic integral equation for $r(t)$ gives

$$r(t) = r(s) + \int_s^t a\,(b - r(u))\,du + \sigma \int_s^t \sqrt{r(s)}\,dZ(u), \quad s \leqslant t. \tag{6}$$

The short rate assumes the normal distribution with mean and variance formula is as follows

$$\mathrm{E}[r(t)|r(s)] = r(s)\,e^{-a(t-s)} + b\big(1 - e^{-a(t-s)}\big), \tag{7}$$

$$\mathrm{var}[r(t)|r(s)] = \frac{r(s)\sigma^2}{a}(e^{-a(t-s)} - e^{-2a(t-s)}) + \frac{b\sigma^2}{2a}(1 - e^{-a(t-s)})^2, \tag{8}$$

respectively.

2.3. The log-likelihood functions

Following [13] the maximum likelihood estimates of the parameters for the both models are found by maximizing the log-likelihood functions of both models. The training sets of KLIBOR will be applied to the Maximum Likelihood Estimation (MLE) method. The log-likelihood function for the Vasicek model is defined as

$$\ell(a, b, \sigma) = -\frac{n}{2}\ln(2\pi) - \frac{1}{2}\sum_{t=1}^{n-1}\left[\ln\left(\frac{\sigma^2}{2a}\big(1 - e^{-2adt}\big)\right) + \frac{(r(t) - r(s)e^{-adt} - b\,(1 - e^{-adt})^2}{\frac{\sigma^2}{2a}\big(1 - e^{-2adt}\big)}\right]. \tag{9}$$

A simplified log-likelihood function, as proposed in [13] for the CIR model is adopted to speed up the calibration process, given in Eq. (10),

$$\ell(a,b,\sigma)=\sum_{t=1}^{n}\left[-\frac{1}{2}\ln\left(2\pi\sigma^2 r(s)\,dt\right)-\frac{\big(r(t)-r(s)-a(b-r(s))\,dt\big)^2}{2\sigma^2 r(s)dt}\right]. \tag{10}$$

The convention of 252 trading days in a year is assumed, that sets a time difference of $dt=\frac{1}{252}$ for the time series data. The selected initial values are $a=0.01$, b is the mean of interest rates from the training data sets, and σ is the volatility of interest rates from the training data sets are fixed for the calibration. In total, there are six sets of parameters (three for the Vasicek model and three for the CIR model) for each type of data set (1-month, 6-month, and 12-month). The estimated parameters, along with 1 000 random numbers drawn from the standard normal distribution, produce 1 000 interest rate paths. Using the simulated error terms and the estimated parameters, $r(t)$ will be estimated using Eq. (2) and Eq. (6) above. The means from the simulations are then compared to the actual rates from the validation sets.

2.4. Estimating the model errors and out-sample forecast accuracies

The goodness-of-fit of two models is tested directly from the value of the maximum likelihood function following the results from [13] as well as the Akaike Information Criterion (AIC) and Bayesian Information Criterion (BIC) scores according to [14]. Equations (11) and (12) are the formulas to calculate AIC and BIC, for each data set, where k is the number of estimated parameters and $\hat{L}$ is the maximized value of each model's log-likelihood function,

$$\mathrm{AIC}=2k-2\ln(\hat{L}), \tag{11}$$

$$\mathrm{BIC}=k\ln(n)-2\ln(\hat{L}). \tag{12}$$

Additionally, the forecast performance of both models is calculated based on residuals between the actual rates, r_i, and the average of the calculated rates, $E(\hat{r}_i)$, obtained from two models for a total of $n=1\,134$ observations for each data set. Following [21], we divide datasets into two parts: the in-sample data sets where the models will be fit into, and out-sample data sets where the forecast errors will be evaluated. The measures of errors chosen are the mean squared error (MSE), root mean squared error (RMSE), and mean absolute percentage error (MAPE). The errors for each data set are measured as follows:

$$\mathrm{MSE}=\frac{1}{n}\sum_{i=1}^{n}\left(r_i-E(\hat{r}_i)\right)^2, \tag{13}$$

$$\mathrm{RMSE}=\sqrt{\mathrm{MSE}}, \tag{14}$$

$$\mathrm{MAPE}=\frac{1}{n}\sum_{i=1}^{n}\left|\frac{r_i-E(\hat{r}_i)}{r_i}\right|\times 100. \tag{15}$$

The model with the lowest errors indicates the most accurate model for future projections of interest rates.

2.5. Estimating the zero-coupon bond prices and yield curve

Results from Section 2.4 will determine which model is the best fit and accurate for estimating future interest rates. This model, along with the best data set, will be adopted to price zero-coupon bonds with maturities of $1,2,5,10,20$ and 30 years and a face value of 100. This would produce a set of discount factors that can be used in valuations and estimations where the uncertainty in the movement of interest rates has been accounted for. Generally, the pricing of a zero-coupon bond at time t that pays 100 at time T where the continuously compounded interest rate at time t is $R(t,T)$ can be defined as $P(t,T)=100\,e^{-R(t,T)\,(T-t)}$. Extending this to the Vasicek and CIR models, the bond price formula can be defined as:

$$P(t,T)=100\,A(t,T)\,e^{-B(t,T)\,r(t)}. \tag{16}$$

Depending on the chosen model, the values of $A(t,T)$ and $B(t,T)$ can be calculated as Eq. (17) and Eq. (18) for the Vasicek model or Eq. (19), Eq. (20) and Eq. (21) for the CIR model,

$$B(t,T)=\frac{1-e^{-a(T-t)}}{a}, \tag{17}$$

$$A(t,T) = \exp\left[\frac{\left(B(t,T) - T + t\right)\left(a^2 b - 0.5\sigma^2\right)}{a^2} - \frac{\sigma^2 B(t,T)^2}{4a}\right], \tag{18}$$

$$B(t,T) = \frac{2\left(e^{\gamma(T-t)} - 1\right)}{(\gamma + a)\left(e^{\gamma(T-t)} - 1\right) + 2\gamma}, \tag{19}$$

$$A(t,T) = \left[\frac{2\gamma e^{(a+\gamma)(T-t)/2}}{(\gamma + a)\left(e^{\gamma(T-t)} - 1\right) + 2\gamma}\right]^{\frac{2ab}{\sigma^2}}, \tag{20}$$

$$\gamma = \sqrt{a^2 + 2\sigma^2} \tag{21}$$

From $P(t,T) = 100\, e^{-R(t,T)\,(T-t)}$, and the calculated $A(t,T)$ and $B(t,T)$ values, the entire term structure of interest rate at time t can be determined as:

$$R(t,T) = -\frac{1}{T-t}\ln A(t,T) + \frac{1}{T-t}\, B(t,T)\, r(t) \tag{22}$$

where $R(t,T)$ represents the yield rate of a zero-coupon bond for a given maturity.

3. Results

3.1. Actual against estimated interest rates

The parameters for each model and data set from Table 1 are adopted into Eq. (2) and Eq. (6) to simulate interest rate paths for the Vasicek and CIR models. The averages from the simulated paths are then compared to the actual rates. The nature of time series models that capture the average of the data seems to over-estimate in the first half of the period and under-estimate in the second half (blue line for the Vasicek model and red line for the CIR model). Table 1 below shows each data set's calibrated parameters of two interest rate models. The graphs of actual against estimated rates are presented in Figure 2 below for two models using three data sets applied to the Vasicek and CIR models.

Table 1. Summary of parameter estimates for the Vasicek and CIR models.

Parameter	Vasicek			CIR		
	1-month	6-month	12-month	1-month	6-month	12-month
a	0.3022	0.1709	0.1685	0.2952	0.1862	0.0345
b	2.9204	3.0466	3.1305	2.9127	3.0752	1.5105
σ	0.2990	0.2769	0.2760	1.0024	0.9107	0.9025

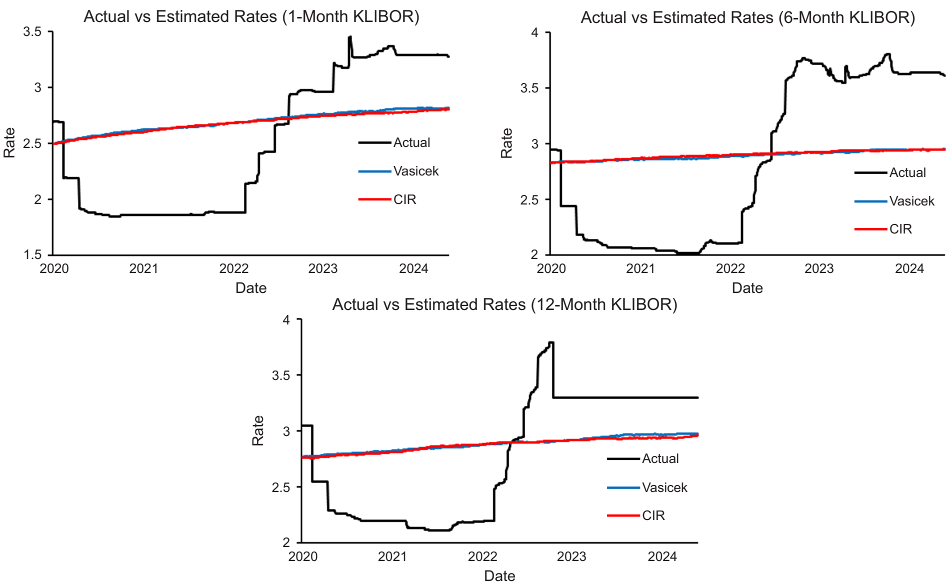

Fig. 2. The comparison of actual and estimated interest rates of 1-, 6-, and 12-month KLIBOR using the Vasicek and CIR models.

3.2. Measures of Goodness-of-Fit and Out-Sample forecast accuracies

The maximized log-likelihood (Log-L) values, along with the AIC and BIC scores for each data, set are calculated to test the goodness-of-fit of the models further. It is found that the Vasicek model consistently surpasses the CIR model, showing higher log-likelihood values and lower AIC and BIC values. Particularly, the Vasicek model using 12-month KLIBOR gives the largest log-likelihood value and the lowest AIC and BIC among the six data sets. The summary of log-likelihood values and AIC and BIC scores is tabulated in Table 2.

Table 2. Summary of goodness-of-fit test for the Vasicek and CIR models.

Parameter	Vasicek				CIR			
	1-month	6-month	12-month	Overall average	1-month	6-month	12-month	Overall average
Log-L	8688	8948	8960	**8865**	8641	8829	8815	8762
AIC	−17370	−17890	−17913	**−17724**	−17276	−17652	−17625	−17518
BIC	−17355	−17875	−17898	**−17709**	−17261	−17637	−17609	−17503

The performance of the models is further analysed based on three measures of errors. The Vasicek model generally provides lower error values and gives lower average values for the three errors. Similarly, the Vasicek model using 12-month KLIBOR again gives the lowest error values compared to the other five data sets. Therefore, the Vasicek model and 12-month KLIBOR are the best samples for pricing zero coupon-bonds and developing a yield curve. These findings support the conclusion from [13]. It is also observed that the main drawback of the Vasicek model did not affect the valuation results when applied to the chosen data. However, it is inconsistent with [14] who concludes that the CIR model is superior to the Vasicek. This is probably due to applying the models to different data sets or validating the models' goodness-of-fit using different methods. In Malaysia, the study by [19] compares the two models using MGS data and concludes that there is no significant difference between them. The summary of the error measures for both models is presented in Table 3 below.

Table 3. Summary of forecast errors for the Vasicek and CIR models.

Measure	Vasicek				CIR			
	1-month	6-month	12-month	Overall average	1-month	6-month	12-month	Overall average
MSE	0.3408	0.5046	0.2598	**0.3684**	0.3481	0.4969	0.3429	0.3960
RMSE	0.5838	0.7103	0.5097	**0.6013**	0.5900	0.7049	0.5856	0.6268
MAPE	24.5571	26.0594	18.6643	**23.0936**	24.8128	26.1544	20.6084	23.8585

3.3. Pricing zero-coupon bonds and estimating the yield curve

Table 4. Average prices and yields of zero-coupon bonds with 100 Face Value.

Maturity	Average Price	Average Yield (%)
1	97.0627	2.9837
2	94.2487	2.9657
5	86.7284	2.8542
10	76.9204	2.6305
20	62.9726	2.3169
30	52.4394	2.1549

The estimated rates from the best sample are used to simulate the prices of zero-coupon bonds for multiple paths using Eq. (17) and Eq. (18). The yield rates of each simulated path are obtained using Eq. (22). Maturities of $1, 2, 5, 10, 20$ and 30 years are chosen to reflect the commonly long period of actuarial liabilities. Results indicate that the prices of short-term bonds exceed those of long-term bonds. These prices are also reflected in the yield-to-maturity for these bonds, where short-term bonds yield more than long-term bonds. Table 4 below displays the average prices and average yields of zero-coupon bonds according to maturity. The average yield rates are used to develop the term structure of interest rates or the yield curve of the zero-coupon bonds. The graph is downward-sloping which indicates an inverted yield curve. This inverted curve confirms that long-term bond yields are lower than short-term bond yields and suggests a possible economic downturn in future years. The same trend is observed for 6-month KLIBOR, however, the yield curve observed using 1-month KLIBOR is flatter. This suggests that the yields for short-term bonds are comparable to those for long-term bonds, making long-term investments less attractive. Figure 3 below shows the respective yield curves based on the average yields from each maturity.

4. Conclusion

This research extended the applications of two continuous-time stochastic interest models, namely the Vasick and CIR models, using KLIBOR data for 1-month, 6-month, and 12-month time frames. Findings suggest that although CIR is the most complex model that considers all four concerns, the Vasicek model is the best model in terms of goodness-of-fit and out-sample forecast accuracies, and particularly worked best for the 12-month rates. In fitting continuous-time interest rate models, the 1-month and 6-month periods might be too short to capture the long-term pattern of interest rates, which makes the 12-month period data the best fitting. The estimated rates from the best model are then used to calculate prices of zero-coupon bonds with maturities of $1, 2, 5, 10, 20$ and 30 years by incorporating uncertainties of future interest rate movements. This step is essential to obtain risk-adjusted discount factors that can be used in valuing long-term actuarial liabilities. The inverted yield curve suggests that long-term bonds do not offer higher yields, that is the opposite fact of what is expected in a normal economy. Further analysis is required to identify whether similar results would be obtained or whether this scenario conforms to four stages of an economic cycle. An extension could include splitting and treating the historical data of KLIBOR during normal and booming economies and recessions separately to understand the impacts of the economic cycle on the models to identify whether it would increase the accuracy of the model.

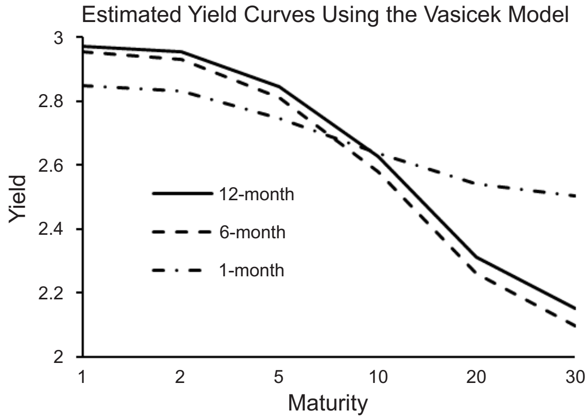

Fig. 3. The Estimated Yield Curves using 1-month, 6-month and 12-month KLIBOR.

[1] Liu C., Sherris M. Immunization and Hedging of Post Retirement Income Annuity Products. Risks. **5** (1), 19 (2017).

[2] Redington F. M. Review of the Principles of Life-office Valuations. Journal of the Institute of Actuaries. **78** (3), 286–340 (1952).

[3] Shiu E. S. W. On the Fisher-Weil immunization theorem. Insurance: Mathematics and Economics. **6** (4), 259–266 (1987).

[4] Shiu E. S. W. Immunization of multiple liabilities. Insurance: Mathematics and Economics. **7** (4), 219–224 (1988).

[5] Shiu E. S. W. On Redington's theory of immunization. Insurance: Mathematics and Economics. **9** (2–3), 171–175 (1990).

[6] Itô K. Stochastic Processes: Lectures given at Aarhus University. Springer Berlin, Heidelberg (2007).

[7] Hull J. Equilibrium Models of the Short Rate. Futures and Other Derivatives (2022).

[8] Kellison S. Stochastic Approaches to Interest. The Theory of Interest (2008).

[9] Vašíček O. An equilibrium characterization of the term structure. Journal of Financial Economics. **5** (2), 177–188 (1977).

[10] Cox J. C., Ingersoll J. E., Ross S. A. A theory of the term structure of interest rates. Econometrica. **53** (2), 385–407 (1985).

[11] Chang Y., Sherris M. Longevity Risk Management and the Development of a Value-Based Longevity Index. Risks. **6** (1), 10 (2018).

[12] Halgašová J., Stehlíková B., Bučková Z. Estimating the Short Rate from the Term Structures in the Vasicek Model. Tatra Mountains Mathematical Publications. **61** (1), 87–103 (2015).

[13] Fergusson K., Platen E. Application of Maximum Likelihood Estimation To Stochastic Short Rate Models. Annals of Financial Economics. **10** (02), 1550009 (2015).

[14] Bao N. K. Q., Hong D. T. T. On the Effect of Interest Rates Dynamics on Vietnamese Companies. International Journal of Economics and Finance. **7** (5), 147–152 (2015).

[15] Sulma S., Widana I. N., Toaha S., Fitria I. Comparison of projected unit credit and entry age normal methods in pension fund Vasicek and Cox–Ingersoll–Ross models. BAREKENG: Jurnal Ilmu Matematika dan Terapan. **17** (4), 2421–2432 (2023).

[16] Ding Y. The Practicality of Vasicek Model in China's Financial Market. SHS Web of Conferences. **163**, 01016 (2023).

[17] Lian K. K. Sectoral Beta Forecasts of Securities in a Thin Capital Market: A Case of Malaysia. Jurnal Pengurusan. **16**, 13–32 (1997).

[18] Nadarajan S., Nur-Firyal R. Comparing Vasicek Model with ARIMA and GBM in Forecasting Bursa Malaysia Stock Prices. Proceedings of the 29th National Symposium on Mathematical Sciences. **2905** (1), 050004 (2024).

[19] Chiet K. W. Pricing of Annuities with Guaranteed Minimum Withdrawal Benefits under Stochastic Interest Rates. Thesis, Universiti Tunku Abdul Rahman (2016).

[20] https://financialmarkets.bnm.gov.my/

[21] Shair S. N., Yusof A. Y., Asmuni N. H. Evaluation of the product ratio coherent model in forecasting mortality rates and life expectancy at births by states. AIP Conference Proceedings. **1842** (1), 030010 (2017).

Моделювання короткострокових процентних ставок за допомогою моделей Васічека та Кокса-Інгерсолла-Росса

Мд Саат А. С.[1], Шаір С. Н.[1,2], Мд Лазам Н.[1,2], Юсоф А. Й.[1], Мохд Амін М. Н.[1,2], Ібрагім Р. І.[3], Мохд Гані Н. А.[1]

[1] *Школа математичних наук, Коледж обчислювальної техніки, інформатики та математики, Технологічний університет MARA (UiTM), 40450, Шах Алам, Селангор*
[2] *Група дослідницьких інтересів актуарного ризику, аналітика та такафул, Університет технологій MARA (UiTM), 40450, Шах Алам, Селангор*
[3] *Факультет науки і технологій, Ісламський науковий університет Малайзії, Бандар Бару Нілай, 71800 Нілай, Негері-Сембілан, Малайзія*

При моделюванні майбутніх невизначеностей ефект часової вартості грошей часто враховується мінімально. Багато моделей передбачають постійні ставки, що призводить до потенційних помилок у ціноутворенні фінансових інструментів. У статті досліджуються неперервні у часі моделі процентних ставок для врахування майбутніх невизначеностей процентних ставок. Прийнято дві стохастичні моделі процентних ставок, Васічека та Кокса–Інгерсолла–Росса (CIR), і оцінено їхню прогнозовану ефективність. Використовуючи оцінку максимальної ймовірності (MLE), моделі підігнані до даних міжбанківської ставки пропозиції Куала–Лумпура (KLIBOR) (1-місячні, 3-місячні та 12-місячні ставки). Згодом було проаналізовано ступінь відповідності та точність прогнозу обох моделей. Результати показують, що модель Васічека є кращою на основі показників AIC, BIC, MSE, RMSE та MAPE. Модель Васічека перевершує CIR, особливо для 12-місячних ставок. Накінець, оцінено ціни облігацій з нульовим купоном і розроблено строкову структуру процентних ставок, виявляючи інвертовану криву прибутковості.

Ключові слова: *стохастичне моделювання; процентний ризик; модель Васічека; модель Кокса–Інгерсола–Росса.*