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Chebyshev approximation by the exponent from a rational expression

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A method for constructing Chebyshev approximation with relative error of the exponential from a rational expression is proposed. It implies constructing an intermediate Chebyshev approximation with absolute error by a rational expression of the logarithm of the function being approximated. The approximation by a rational expression is calculated as the boundary mean-power approximation using an iterative scheme based on the least squares method with two variable weight functions. The presented results of solving test examples confirm the fast convergence of the method.

Keywords: *Chebyshev approximation by exponential expression; rational expression; mean-power approximation; least squares method; variable weight function.*

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1. Problem statement

Let us approximate a continuous positive function $f(x)$ on the interval $[\alpha,\beta]$ with relative error by an exponential expression

$$E_{k,l}(a,b;x) = e^{R_{k,l}(a,b;x)}, \quad k \geqslant l, \tag{1}$$

where $R_{k,l}(a,b;x)$ is a rational expression

$$R_{(k,l)}(a,b;x) = \frac{\sum_{i=0}^{k} a_i x^i}{1+\sum_{i=1}^{l} b_i x^i}, \quad k \geqslant l \tag{2}$$

and a_i, $i=\overline{0,k}$, b_i, $i=\overline{1,l}$ are the unknown parameters, $\{a_i\}_{i=0}^{k} \in A$, $A \subseteq \mathbb{R}^{k+1}$, $\{b_i\}_{i=1}^{l} \in B$, $B \subseteq \mathbb{R}^{l}$, $\mathbb{R}^n$ is an n-dimensional vector space. Exponential dependencies are used in approximating mathematical and special functions [1], modeling physical [2–5], chemical [6], biological, and other processes [7–9], as well as in designing measurement systems and automatic control systems [10–14].

The expression $E_{(k,l)}(a,b;x)$ will be called a Chebyshev approximation of the function on the interval with relative error if it satisfies the condition

$$\max_{x\in[\alpha,\beta]} \left|1-\frac{E_{(k,l)}(a^*,b^*;x)}{f(x)}\right| = \min_{a\in A, b\in B}\left(\max_{x\in[\alpha,\beta]}\left|1-\frac{E_{(k,l)}(a,b;x)}{f(x)}\right|\right). \tag{3}$$

Research on the properties of Chebyshev approximation by nonlinear expressions of polynomials and other expressions has been the subject of many works, including [15–19]. Efficient methods for constructing Chebyshev approximation by some nonlinear expressions of polynomials have been proposed in works [17–19]. In general, constructing Chebyshev approximation by nonlinear expressions of polynomials and rational polynomials is a complex problem of nonlinear optimization [15–17]. We propose a method for constructing Chebyshev approximation by the exponential of a rational expression (1), which involves using an intermediate Chebyshev approximation by a rational expression. The Chebyshev approximation by a rational expression $R_{(k,l)}(a,b;x)$ is calculated as the boundary approximation in the norm of space L^p as $p\to\infty$ using the method described in works [20, 21].

2. Properties of Chebyshev approximation with relative error by the exponential of a rational expression

For simplicity of presentation, the rational expression (2) is given in the form

$$R_{(k,l)}(a,b;x) = a_0 + \frac{1+\sum_{i=1}^{k}\bar{a}_i x^i}{1+\sum_{i=1}^{l} b_i x^i} = a_0 + \bar{R}_{(k,l)}(\bar{a},b;x), \tag{4}$$

where

$$\bar{R}_{(k,l)}(\bar{a},b;x) = \frac{1+\sum_{i=1}^{k}\bar{a}_i x^i}{1+\sum_{i=1}^{l} b_i x^i}, \quad \bar{a}_i = \begin{cases} a_i - a_0 b_i, & \text{if} \quad i=\overline{1,l}, \\ a_i, & \text{if} \quad i=\overline{l+1,k}. \end{cases} \tag{5}$$

The exponential expression (1) using the representation of the rational expression in the form (5) will be

$$E_{(k,l)}(a,b;x) = e^{a_0+\bar{R}_{(k,l)}(\bar{a},b;x)} = \bar{a}_0 e^{\bar{R}_{(k,l)}(\bar{a},b;x)}, \qquad \bar{a}_0 = e^{a_0}. \tag{6}$$

The construction of Chebyshev approximation by an exponential expression (6) is a complex problem of nonlinear optimization [15, 20]. The possibility of linearizing the problem of constructing Chebyshev approximation with relative error by the exponent of a rational expression (1) is based on Theorem 1.

Theorem 1. *Let the function $f(x)$ be positive and continuous on the interval $[\alpha,\beta]$. Chebyshev approximation by an exponential expression (1) of the function $f(x)$ with relative error on the interval $[\alpha,\beta]$ is defined through the Chebyshev approximation by a rational expression (4) of the function $f_l(x)=\ln(f(x))$ on $[\alpha,\beta]$ with absolute error. The values of parameters $\bar{a}_i$, $i=\overline{1,k}$ and b_i, $i=\overline{1,l}$ of the approximation by the exponential expression in the form of (6) coincide with the values of the same-name parameters in the approximation by the rational expression $\bar{R}_{(k,l)}(\bar{a},b;x)$ (5), and the value of the parameter $\bar{a}_0$ is calculated by the formula*

$$\bar{a}_0 = \frac{2f(x_{\max})\,f(x_{\min})}{\bar{E}_{(r,l)}(\bar{a},b;x_{\min})\,f(x_{\max}) + \bar{E}_{(r,l)}(\bar{a},b;x_{\max})\,f(x_{\min})}, \tag{7}$$

where $x_{\max}$ is the point where the relative error of the function $f(x)$ approximation by expression

$$\bar{E}_{(k,l)}(\bar{a},b;x) = e^{\bar{R}_{(k,l)}(\bar{a},b;x)} \tag{8}$$

achieves its maximum value on the interval $[\alpha,\beta]$, and $x_{\min}$ is the point where the value of the relative error is minimized.

Proof. According to the characteristic property of Chebyshev approximation [15,22], for the existence of the approximation of the function $f(x)$ by expression (1) with relative error on the interval $[\alpha,\beta]$ it is sufficient for the system of equations

$$1 - \frac{\bar{a}_0 e^{\bar{R}_{(k,l)}(\bar{a},b;z_j)}}{f(z_j)} = (-1)^j \mu, \quad j=\overline{1,k+l+2} \tag{9}$$

to have a unique solution regarding the unknown parameters $\bar{a}_i$, $i=\overline{0,k}$ and b_i, $i=\overline{1,l}$, and the error μ, where z_j, $j=\overline{1,k+l+2}$, are the points of Chebyshev alternance on the interval $[\alpha,\beta]$ in ascending order.

By sequentially subtracting the (j)th equation from the $(j+2)$th equation of the system (9), $j=\overline{1,k+l}$, we eliminate the unknowns $\bar{a}_0$ and μ from the system of equations (9)

$$\frac{e^{\bar{R}_{(k,l)}(\bar{a},b;z_{j+2})}}{f(z_{j+2})} = \frac{e^{\bar{R}_{(k,l)}(\bar{a},b;z_j)}}{f(z_j)}, \quad j=\overline{1,k+l}.$$

Since, according to the theorem condition, the function $f(x)$ is positive, this system of equations is equivalent to the system

$$\bar{R}_{(k,l)}(\bar{a},b;z_{j+2}) - \bar{R}_{(k,l)}(\bar{a},b;z_j) = \ln(f(z_{j+2})) - \ln(f(z_j)), \quad j=\overline{1,k+l}. \tag{10}$$

In the work [15], it is established that the alternance points of the Chebyshev approximation by expression (4) coincide with the alternance points by expression $\bar{a}_0 e^{\bar{R}_{(k,l)}(\bar{a},b;x)}$ in (6). Taking into account the property established in [15], we assert that the system of equations (10) coincides with the system of equations for determining the parameters $\bar{a}_i$, $i=\overline{1,k}$ and b_i, $i=\overline{1,l}$ of the Chebyshev approximation with absolute error of the function $f_l(x)$ by a rational expression in the form (4). Based on the characteristic condition for the existence of Chebyshev approximation with absolute error of the function $f_l(x)$ by a rational expression (4) [15,22], its parameters satisfy the system of equations:

$$f_l(z_j) - a_0 - \bar{R}_{(k,l)}(\bar{a}, b; z_j) = (-1)^j \bar{\mu}, \quad j=\overline{1,k+l+2}, \tag{11}$$

where z_j, $j=\overline{1,k+l+2}$ are the points of Chebyshev alternance on the interval $[\alpha,\beta]$ in ascending order, which coincide with the corresponding points of system (9), $\bar{\mu}$ is the error of approximation, and $f_l(x)=\ln(f(x))$. Similarly to the elimination of unknowns $\bar{a}_0$ and μ from the system of equations (8), let us eliminate the unknowns a_0 and $\bar{\mu}$ from the system of equations (11). After removing these unknowns, we obtain a system of equations that coincides with the system of equations (10).

Since the system of equations (10) for determining the values of parameters $\bar{a}_i$, $i=\overline{1,k}$ and b_i, $i=\overline{1,l}$ of the Chebyshev approximation with relative error by an exponential expression (6) of the function $f(x)$ on the interval $[\alpha,\beta]$ coincides with the system of equations (11) regarding the values of parameters $\bar{a}_i$, $i=\overline{1,k}$ and b_i, $i=\overline{1,l}$ of the Chebyshev approximation of the function $f_l(x)$ by a rational expression (4), calculating the values of these approximation parameters by the exponential expression (1) reduces to calculating the values of the corresponding parameters of the Chebyshev approximation with absolute error of the function $f_l(x)$ by a rational expression (4) on the interval $[\alpha,\beta]$.

The value of the parameter $\bar{a}_0$ can be determined as the solution to a one-parameter Chebyshev approximation problem with relative error of the function $f(x)$ on the interval $[\alpha,\beta]$ by the expression $\bar{a}_0\bar{E}_{(k,l)}(\bar{a},b;x)$, where $\bar{E}_{(k,l)}(\bar{a},b;x)$ is defined by formula (8)

$$\max_{x\in[\alpha,\beta]}\left|1-\frac{\bar{a}_0 e^{\bar{R}_{(k,l)}(\bar{a},b;x)}}{f(x)}\right| \xrightarrow[\bar{a}_0]{} \min \tag{12}$$

The solution to problem (12) regarding the value of parameter $\bar{a}_0$ is calculated by formula (7) [22].

Therefore, the values of parameters $\bar{a}_i$, $i=\overline{1,k}$ and b_i, $i=\overline{1,l}$ of the Chebyshev approximation with relative error by an exponential expression (1) coincide with the values of the same-name parameters of the Chebyshev approximation with absolute error of the function $f_l(x)$ by a rational expression (4) on the interval $[\alpha,\beta]$, and the value of parameter $\bar{a}_0$ is calculated according to formula (7). ∎

3. Method for computing parameters of Chebyshev approximation with relative error by an exponential of a rational expression

According to Theorem 1, constructing the Chebyshev approximation of the function $f(x)$ by an exponential expression (1) with relative error reduces to calculating the parameters of the Chebyshev approximation of the function $f_l(x)$ by a rational expression (4) with absolute error. By satisfying the condition $k \geqslant l$, the rational expression (2) can be represented in the form of (4), and accordingly, the exponential expression (1) will take the form of the product of the exponential function and a constant $\bar{a}_0$ (6). The value of the constant $\bar{a}_0$ is calculated according to Theorem 1 using formula (7). However, during the construction of the Chebyshev approximation of the function $f(x)$ by an exponential expression (1) with relative error, it is not necessary to bring the rational expression to the form (4). The approximation by an exponential expression (1) can be obtained from the Chebyshev approximation with absolute error of the function $f_l(x)$ by a rational expression in the form of (2). In this case, the approximation by an exponential expression (1) will be

$$\tilde{a} e^{R_{(k,l)}(a,b;x)}, \tag{13}$$

where $R_{(k,l)}(a,b;x)$ is the Chebyshev approximation of function $f_l(x)$ by rational expression (2) with absolute error. Value of parameter $\tilde{a}$ is determined as the solution to a one-parameter problem of Chebyshev approximation with relative error of function $f(x)$ by expression $\tilde{a}e^{R_{(k,l)}(a,b;x)}$ on the interval $[\alpha,\beta]$

$$\max_{x\in[\alpha,\beta]}\left|1-\frac{\tilde{a}e^{R_{(k,l)}(a,b;x)}}{f(x)}\right| \underset{\tilde{a}}{\longrightarrow} \min \tag{14}$$

regarding parameter $\tilde{a}$.

The proposed method of constructing Chebyshev approximation by an exponential expression (1) in the form of (13) with the value of parameter $\tilde{a}$ being the solution to problem (14) is equivalent to constructing the approximation in the form of (8) with the determination of $\bar{a}_0$ from (12). By utilizing the method of calculating the parameters of the approximation by the exponential expression (1) in the form of (13), we reduce the computational workload.

We will describe the method of constructing the Chebyshev approximation using the exponential expression (1) for the case of a discrete function on a set of points $X=\{x_i\}_{i=1}^m$, $X\in[\alpha,\beta]$, where $m\geqslant k+l+2$. Calculating the parameters of the Chebyshev approximation of the function $f_l(x)$ by a rational expression (2) can be implemented using the method described in works [20, 21]. This method involves sequentially constructing mean-power approximations using iterations based on the least squares method

$$\sum_{i=1}^{m}\rho_r(x_i)\left(f_l(x_i)-R_{(k,l)}(a,b;x_i)\right)^2 \underset{a\in A,\, b\in B}{\longrightarrow} \min, \quad r=0,1,\ldots,p-2 \tag{15}$$

with sequential refinement of the weight function $\rho_r(x)$ value

$$\rho_0(x)=1, \quad \rho_r(x)=\rho_{r-1}(x)|\Delta_r(x)|=\prod_{i=1}^{r}|\Delta_i(x)|, \quad r=1,\ldots,p-2, \quad p=3,4,\ldots, \tag{16}$$

where $\Delta_s(x)=f_l(x)-R_{(k,l,s-1)}(a,b;x)$, $s=\overline{1,r}$, $R_{(k,l,s)}(a,b;x)$ is the approximation of the function $f_l(x)$ by the least squares method with the weight function $\rho_s(x)$. Approximation $R_{(k,l,s)}(a,b;x)$ corresponds to the mean-power approximation of power $p=s+2$.

Constructing the approximation by a rational expression using the least squares method is a nonlinear problem [23–25]. To construct such an approximation, linearization with variable weight function has been applied [20, 21], which involves iterative refinement of the approximation by a rational expression (2). According to this linearization method, for each fixed value of p, we calculate the approximation of the function $f_l(x)$ by a rational expression $R_{(k,l)}(a,b;x)$ (1) using the least squares method

$$\sum_{i=1}\rho_r(x_i)\upsilon_{r,t}(x_i)\left(\Phi_{r,t}(a,b;x_i)\right)^2 \underset{a\in A,\, b\in B}{\longrightarrow} \min, \quad r=p-2, \quad t=0,1,\ldots, \tag{17}$$

where

$$\Phi_{r,t}(a,b;x)=f_l(x)\left(1+\sum_{i=1}^{l}b_{i,r,t}x^i\right)-\sum_{i=0}^{k}a_{i,r,t}x^i.$$

The value of weight function $\rho_r(x)$ is calculated by formula (16), and the value of weight function $\upsilon_{r,t}(x)$ — by the formula

$$\upsilon_{r,t}(x)=\begin{cases}1, & \text{if} \quad r=0,\ t=0,\\ \left(1+\sum_{i=1}^{l}b_{i,r,t-1}x^i\right)^{-2}, & \text{if} \quad t>0.\end{cases} \tag{18}$$

Refinement of approximation by rational expression (2) using the iterative scheme (17)–(18) can be monitored by accuracy ε_1 of satisfying the condition

$$|\eta_{r,t-1}-\eta_{r,t}|\leqslant\varepsilon_1\eta_{r,t}, \tag{19}$$

where

$$\eta_{r,t}=\sum_{i=1}^{m}\rho_r(x_i)\upsilon_{r,t}(x_i)\left(\Phi_{r,t}(a,b;x_i)\right)^2. \tag{20}$$

During testing, a value of $\varepsilon_1 = 0.003$ was used to ensure convergence of two or three significant digits of the sum of squares of deviations (20) on the interval $[\alpha, \beta]$. Satisfying condition (19) implies that the mean-power approximation of power $p = s + 2$ by a rational expression $R_{(k,l,r)}(a, b; x)$ was calculated with accuracy ε_1. The values of the approximation parameters $R_{(k,l,r)}(a, b; x)$ are the following:

$$a_{j,r} = a_{j,r,t} \quad (j = \overline{0, k}), \quad b_{j,r} = b_{j,r,t} \quad (j = \overline{1, l}).$$

Therefore, constructing the Chebyshev approximation by a rational expression (2) involves two iterative processes: nested iterations (17)–(18) and outer iterations (15)–(16). The completion of iterations (15)–(16) is monitored by achieving some prescribed accuracy ε

$$\mu_{r-1} - \mu_r \leqslant \varepsilon \mu_r, \tag{21}$$

where

$$\mu_r = \max_{1 \leqslant i \leqslant m} \left| f_l(x_i) - R_{(k,l,r)}(a, b; x_i) \right|.$$

As a result of iterations (15)–(16), we obtain the Chebyshev approximation with absolute error of the function $f_l(x)$ by a rational expression $R_{(k,l)}(a, b; x)$ on the interval $[\alpha, \beta]$ with accuracy ε. The Chebyshev approximation of the function $f(x)$ by an exponential expression (1) is determined according to expression (13), where the value of parameter $\tilde{a}$ is calculated by the formula

$$\tilde{a} = \frac{2 f(x_{\max})\, f(x_{\min})}{\tilde{E}_{(r,l)}(a, b; x_{\min})\, f(x_{\max}) + \tilde{E}_{(r,l)}(a, b; x_{\max})\, f(x_{\min})}, \tag{22}$$

where $x_{\max}$ is the point at which the relative error of the approximation of the function $f(x)$ by the expression

$$\tilde{E}_{(k,l)}(a, b; x) = e^{R_{(k,l)}(a,b;x)}$$

reaches its maximum value on the set of points $X = \{x_i\}_{i=1}^m$, and $x_{\min}$ is the point where the value of the relative error is minimized. The value of parameter $\tilde{a}$ (22) is obtained as the solution to the one-parameter problem (14) of Chebyshev approximation with relative error of the function $f(x)$ by expression $\tilde{E}_{(k,l)}(a, b; x)$ on the on the set of points X.

The results of calculating the parameters of the Chebyshev approximation with relative error by an exponential expression (1) for test examples confirm the fast convergence of the described method.

Example 1. Find the Chebyshev approximation with a relative error by the exponential expression

$$E_{2,1}(a, b; x) = \exp\left(\frac{a_0 + a_1 x + a_2 x^2}{1 + b_1 x}\right) \tag{23}$$

of the function

$$y(x) = \exp\left(\frac{1.57 - 12x + 11.75x^3}{1 + 12.5x + 3.75x^2 + 0.73x^3}\right),$$

given at the points x_i, $i = \overline{0, 30}$, where $x_i = 0.1i$.

Using the proposed method for $\varepsilon = 0.003$ in condition (21), in 8 iterations (15)–(16) for the function

$$y_l(x) = \frac{1.57 - 12x + 11.75x^3}{1 + 12.5x + 3.75x^2 + 0.73x^3},$$

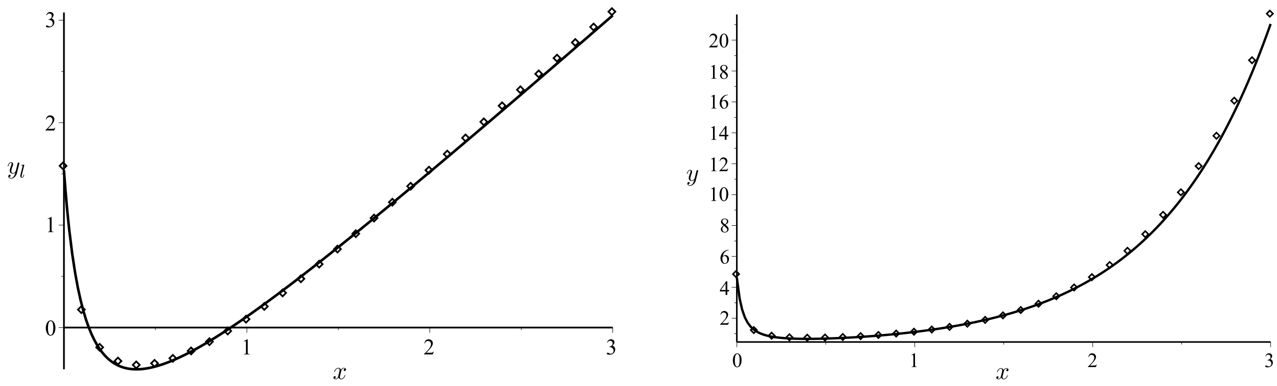
we have obtained Chebyshev approximation by the rational expression

$$R_{2,1}(a, b; x) = \frac{1.59440958x^2 - 1.670827822x + 0.1976708639}{0.1288905399 + x}. \tag{24}$$

This rational expression provides the absolute approximation error of the function $y_l(x)$ of 0.03788. Accordingly, the Chebyshev approximation of the function $y(x)$ by the exponential expression

$$E_{2,1}(a, b; x) = 0.9992828186 \exp\left(\frac{1.59440958x^2 - 1.670827822x + 0.1976708639}{0.1288905399 + x}\right) \tag{25}$$

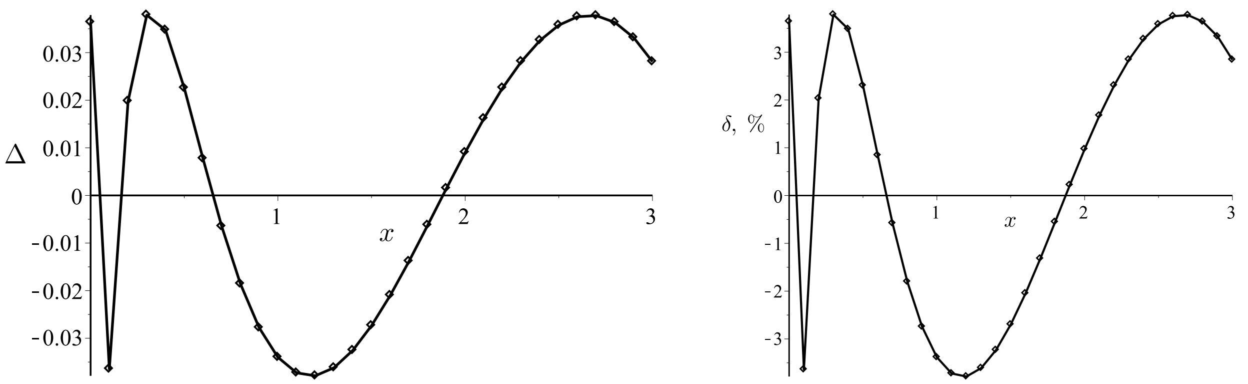
provides the relative approximation error of 3.78661%. The graphs of the functions $y_l(x)$ and $y(x)$ are shown in Figure 1.



a ***b***

Fig. 1. The graphs of: (***a***) function $y_l(x)$, (***b***) function $y(x)$.

The specified values of the function we are approximating are depicted by points in Figure 1, while solid lines represent the obtained approximations. The curves of the approximation errors (24) and (25) are shown in Figure 2.



a ***b***

Fig. 2. The graphs of (***a***) absolute error of approximation of function $y_l(x)$ by rational expression (24), (***b***) relative error of approximation of function $y(x)$ by exponential expression (25).

The error curves shown in Figure 2 correspond to the characteristic property of Chebyshev approximation. They have five alternance points where the absolute error values of the approximation converge within the specified accuracy, and the sign of the deviation alternates at these points [22].

In the extremum points, the error of approximation (25) takes on the following values:

$$\begin{gathered}(0, 0.03640464601), \quad (0.1, -0.03639154382), \quad (0.3, 0.03786619223),\\ (1.2, -0.03786619231), \quad (2.7, 0.03775122311)\end{gathered} \tag{26}$$

The divergence of error values at alternance points (26) can be reduced by increasing the accuracy of calculation ε in condition (14). Chebyshev approximation of the function $y(x)$ by exponential expression $E_{2,1}(a, b; x)$ for $\varepsilon = 0.00003$

$$E_{2,1}(a, b; x) = 0.1291867109 \exp\left(\frac{11.594953693x^2 - 1.671977115x + 0.1980107527}{0.1291867109 + x}\right) \tag{27}$$

obtained in 29 iterations with the error of 3.759%. The divergence of the absolute values of the approximation error (27) in the points of alternance:

$$\begin{gathered}(0, 0.03724707567), \quad (0.1, -0.03717217393), \quad (0.3, 0.03750955719),\\ (1.2, -0.0375922501), \quad (2.7, 0.03759224965)\end{gathered}$$

has decreased.

4. Conclusions

The method of constructing Chebyshev approximation with relative error for a tabular function by an exponent from a rational expression has been implemented using intermediate Chebyshev approximation by a rational expression with absolute error of logarithm values of the approximated function. The Chebyshev approximation by a rational expression is calculated as the boundary mean-power approximation based on the least squares method with two variable weight functions. The proposed method allows for constructing Chebyshev approximation by an exponential expression with the required accuracy. The results of test examples confirm the rapid convergence of the method.

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Чебишовське наближення експонентною від раціонального виразу

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Запропоновано метод побудови чебишовського наближення з відносною похибкою експонентою від раціонального виразу. Він полягає в побудові проміжного чебишовського наближення з абсолютною похибкою раціональним виразом значень логарифму функції, що наближається. Наближення раціональним виразом обчислюється як граничне середньостепеневе наближення за ітераційною схемою на основі методу найменших квадратів з двома змінними ваговими функціями. Представлені результати розв'язування тестових прикладів підтверджують швидку збіжність методу.

Ключові слова: *чебишевська апроксимація експоненціальним виразом; раціональне вираження; апроксимація середньої потужності; метод найменших квадратів; функція змінної ваги.*