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Numerical solution of the vertical infiltration problem in bounded profiles

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There is presented a numerical solution of the one-dimensional infiltration problem in bounded profiles. The soil is assumed to have constant water diffusivity and linear dependence between the hydraulic conductivity and the water content. Then, the vertical infiltration problem is modeled as an initial boundary value problem for a diffusion equation. We combine the finite difference scheme for the time variable with the fundamental sequence method for the spatial variable. The derived numerical scheme is applied to both flooding and rainfall scenarios. The convergence of the numerical approximated solution to the analytical one justifies the applicability of the method.

Keywords: *advection-dispersion equation; Rothe's method; fundamental sequences method.*

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1. Introduction

The problem of infiltration arises when water, from irrigation or rainfall, reaches the soil surface and begins to penetrate the soil. Understanding this process is crucial for various applications, including agriculture, environmental science, and water resources management. The mathematical modeling of infiltration involves describing the complex interactions between soil properties (water diffusivity D_0 and hydraulic conductivity K_0) and the dynamics of water movement within the soil profile. The fundamental PDE that describes the movement of water through soil is the Richards' equation. However due to its non-linear nature, further assumptions on the soil properties (constant diffusivity and K_0 linear function of water content) are added to reduce it to the so-called advection-dispersion equation, a diffusion-type PDE [1, 2]. This PDE is considered to model the problem of vertical infiltration in bounded profiles, meaning the one-dimensional problem in a bounded interval.

Both agricultural engineers and applied mathematicians are studying this problem providing both analytical and numerical solutions. The analytical solutions are limited and mainly "approximate" since the complementary error function appears in the formulas. We refer to [3] and [4] where approximate analytical solutions were presented, using the Laplace transform, for the advection-dispersion equation for flooding and rainfall with constant flux, respectively. See also the review paper [5]. Recently the Fokas method was applied to derive an analytical solution as an integral representation evolving the initial and boundary data for bounded [6] and semi-infinite [7] profiles.

On the other hand, numerical methods, particularly finite difference, and finite element methods, have become essential tools for solving the problem of vertical infiltration. These methods allow for the simulation of soil-water interactions under various conditions, see for example [8–10]. In the last years, new and advanced numerical schemes were proposed, like adaptive [11], meshless [12] and spectral methods [13].

In this direction, we propose to solve the infiltration problem numerically using a two-step approach. First, the time-dependent problem is reduced to a sequence of the stationary problems using the Rothe's method [14]. For the second step, where the sequence of elliptic problems is fully discretized, we consider the Fundamental Sequences Method (FSM). The FSM is inspired by the Method of Fundamental Solutions (MFS), being a popular choice for solving unsteady problems having a known explicit fundamental solution, see [15–17]. In the MFS, the unknown function is approximated by a linear combination of the fundamental solutions followed by collocation on the boundary for finding the unknown coefficients in the expansion. The FSM was firstly introduced in [18] for a sequence of elliptic problems with the recurrent right side, obtained after time discretization of the parabolic and hyperbolic problems. In this case, the unknown function is approximated by the linear combination of the functions from the known fundamental sequence.

An outline of the paper is: in section 2 we state the one-dimensional infiltration problem, as well as substitution for avoiding the first derivative of the spatial variable. Time discretization using the Rothe's method is given in section 3. Obtained sequence of elliptic equations is fully discretized using the FSM in section 4. The results of numerical examples for both flooding and rainfall problems are given in section 5.

2. Problem statement

There is considered the problem of finding the solution θ (water content) satisfying

$$\frac{\partial \theta}{\partial t}(x,t) + K_0 \frac{\partial \theta}{\partial x}(x,t) = D_0 \frac{\partial^2 \theta}{\partial x^2}(x,t), \quad (x,t) \in (0,L) \times (0,T], \tag{1a}$$

$$\theta(x,0) = \theta_0, \qquad x \in (0,L), \tag{1b}$$

$$\theta(0,t) = f(t), \quad \theta(L,t) = g(t), \qquad t \in (0,T], \tag{1c}$$

where $K_0, D_0 > 0$, f, g are given smooth functions, $\theta_0 \in \mathbb{R}$ and $T > 0$ is the final time. Equation (1a) describes the water propagation in a bounded soil of length $L > 0$ for all positive times t (T can be arbitrary large). The soil is characterized by the water diffusivity D_0 and the hydraulic conductivity K_0. For flooding in the surface $x = 0$ we set f to be constant and if f is time-dependent we model rainfall. In the following examples, g will be either constant or zero.

In order to avoid the first derivative of the spatial variable in (1a) and obtain homogeneous initial condition, the following substitution is applied

$$\theta(x,t) = e^{\frac{K_0}{2D_0}x} u(x,t) + \theta_0, \quad (x,t) \in (0,L) \times (0,T]. \tag{2}$$

As a result, the initial boundary value problem for u is obtained as

$$\frac{1}{D_0}\frac{\partial u}{\partial t}(x,t) + \frac{K_0^2}{4D_0^2} u(x,t) = \frac{\partial^2 u}{\partial x^2}(x,t), \quad (x,t) \in (0,L) \times (0,T], \tag{3a}$$

$$u(x,0) = 0, \qquad x \in (0,L), \tag{3b}$$

$$u(0,t) = f(t) - \theta_0, \qquad t \in (0,T], \tag{3c}$$

$$u(L,t) = \big(g(t) - \theta_0\big) e^{-\frac{K_0}{2D_0}L}, \qquad t \in (0,T]. \tag{3d}$$

We solve (3) in two steps. First we perform a discretization with respect to t and then apply the FSM. Then, the approximated u is substituted into (2) to get the value of the water content θ.

3. Semi-discretization with respect to time

Following [14], we reduce (3) to a sequence of stationary problems using Rothe's method. Time derivative in the equation (3a) is approximated by the backward Euler difference approximation on the equidistant mesh

$$t_n = (n+1)h, \quad \text{for} \quad n = -1, 0, \ldots, N-1, \quad h = \tfrac{T}{N}, \quad N \in \mathbb{N}. \tag{4}$$

The solution $u(\cdot, t_n)$ is approximated by a sequence u_n, $n = 0, \ldots, N-1$ and the elements of this sequence satisfy the boundary value problems

$$u_n''(x) - \gamma^2 u_n(x) = \beta u_{n-1}(x), \quad x \in (0,L), \tag{5a}$$

$$u_n(0) = \tilde{f}_n, \quad u_n(L) = \tilde{g}_n, \tag{5b}$$

for $n = 0, \ldots, N-1$, with $u_{-1} = 0$, $\gamma^2 = \frac{K_0^2}{4D_0^2} + \frac{1}{D_0 h}$, $\beta = -\frac{1}{D_0 h}$, $\tilde{f}_n = f(t_n) - \theta_0$ and $\tilde{g}_n = \big(g(t_n) - \theta_0\big) e^{-\frac{K_0}{2D_0} L}$.

An alternative approach for time discretization could be the application of the Laguerre transform, see for example [18]. In the next section the FSM is applied for the discretization of (5).

4. Fundamental sequences method for numerical solution of the stationary problems

We define the fundamental sequence of the elliptic boundary value problems (5).

Definition 1. *The sequence of functions Φ_n, $n = 0, 1, \ldots, N$ is the fundamental sequence for the equations in (5a), provided that*

$$\frac{\partial^2 \Phi_n}{\partial x^2}(x, y) - \gamma^2 \Phi_n(x, y) - \beta \Phi_{n-1}(x, y) = \delta(x - y), \quad x \neq y, \tag{6}$$

where δ is the Dirac delta function and $\Phi_{-1} = 0$.

In [19] the explicit representation of the elements in the fundamental sequence was found and here we recall the result.

Theorem 1. *The sequence of functions Φ_n with*

$$\Phi_n(x, y) = e^{-\gamma |x-y|}\, v_n(|x - y|), \quad x \neq y, \tag{7}$$

for $n = 0, 1, \ldots, N$ is a fundamental sequence of the elliptic equations (5a) in the sense of Definition 1.

The polynomials v_n for $n = 0, 1, \ldots, N$ are given by

$$v_n(r) = \sum_{m=0}^{p} a_{n,m} r^m, \tag{8}$$

with

$$\begin{aligned} &a_{n,0} = 1, && n = 0, \ldots, N, \\ &a_{n,n} = -\frac{1}{2\gamma n}\, \beta a_{n-1,n-1}, && n = 1, \ldots, N, \\ &a_{n,k} = \frac{1}{2\gamma k}\big(k(k+1)\, a_{n,k+1} - \beta\, a_{n-1,k-1}\big), && n = 2, \ldots, N,\ k = n-1, \ldots, 1. \end{aligned}$$

As in [18, 19] the solution to (5) is approximated using a linear combination of functions from the fundamental sequence

$$u_n(x) \approx \tilde{u}_n(x) = \sum_{m=0}^{n} \sum_{k=1}^{2} \alpha_{m,k}\, \Phi_{n-m}(x, y_k), \quad x \in (0, L), \tag{9}$$

with given source points y_1 and y_2, located outside the $(0, L)$, i.e. $y_1 < 0$ and $y_2 > L$, and unknown coefficients $\alpha_{n,k}$, $n = 0, \ldots, N-1$, $k = 1, 2$. By direct differentiation, it can be verified that the approximation (9) satisfies (5a).

Collocating to match the boundary data in (5b), we generate a recurrent sequence of linear systems for $n = 0, 1, \ldots, N-1$ to find the coefficients $\alpha_{n,k}$

$$\begin{cases} \displaystyle\sum_{k=1}^{2} \alpha_{n,k}\, \Phi_0(0, y_k) = \tilde{f}_n - \sum_{m=0}^{n-1} \sum_{\ell=1}^{2} \alpha_{m,\ell}\, \Phi_{n-m}(0, y_\ell), \\ \displaystyle\sum_{k=1}^{2} \alpha_{n,k}\, \Phi_0(L, y_k) = \tilde{g}_n - \sum_{m=0}^{n-1} \sum_{\ell=1}^{2} \alpha_{m,\ell}\, \Phi_{n-m}(L, y_\ell). \end{cases} \tag{10}$$

The systems (10) consist of the same 2×2 matrix and recurrent right-hand side vectors, this is obtained from the observation that only the coefficients $\alpha_{n,1}$ and $\alpha_{n,2}$ in front of Φ_0 have not been previously used in (9).

Having found solutions to (10), taking into account (2) and (9), we can build the numerical approximation to the solution of the problem (1) at the mesh points $t_n, n = 0, \dots, N-1$ (4) by

$$\theta(x, t_n) \approx e^{\frac{K_0}{2D_0}x}\, \tilde{u}_n(x) + \theta_0, \quad x \in (0, L). \tag{11}$$

5. Numerical examples

In this section we examine the effectiveness and convergence of the numerical solution (11) by comparing it with given analytical solution. We consider two cases: flooding and rainfall.

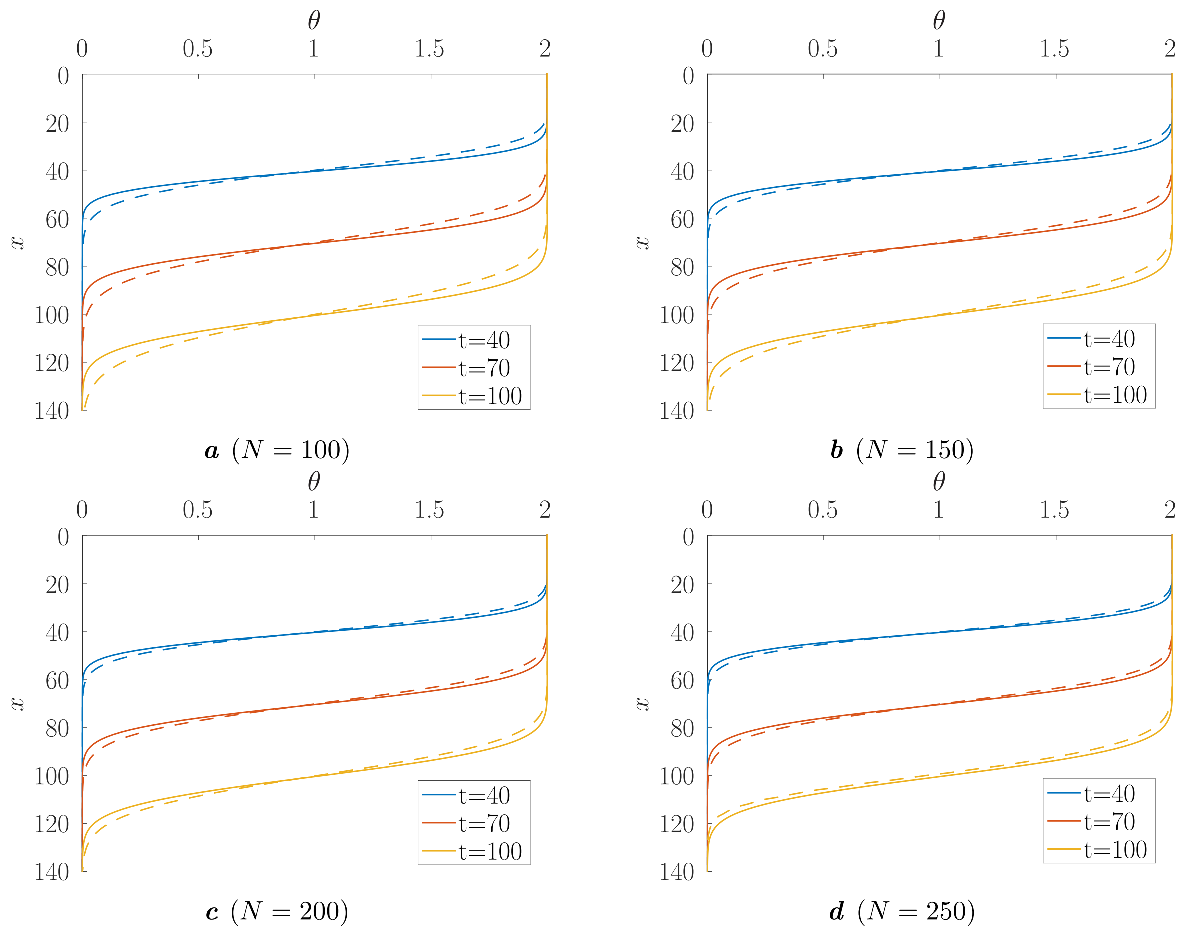
5.1. Flooding

In the first example, we assume constant boundary functions $f(t) = \theta_1$ and $g(t) = \theta_0$. We set $\theta_1 = 2$ and $\theta_0 = 0$ the soil water content at saturation and initially, respectively. An analytical solution was derived in [20] for the specific case of $D_0 = 0.5$ and $K_0 = 1$. Then, the series representation of the exact solution is given by

$$\theta(x,t) = \left(\theta_1 - \theta_0\right) e^x \left[\frac{\sinh(L-x)}{\sinh L} + \frac{2\pi}{L^2}\sum_{n=1}^{\infty} \frac{(-1)^n n \sin\left(n\pi \frac{L-x}{L}\right)}{1 + \frac{n^2\pi^2}{L^2}} e^{-\left(1+\frac{n^2\pi^2}{L^2}\right)\frac{t}{2}}\right] + \theta_0. \tag{12}$$

The final time is chosen as $T = 100$ and the length $L = 140$. According to [21], the source points should be located not too close and not too far from the endpoints of the interval $(0, L)$, so we choose $y_1 = -1$ and $y_2 = L + 1$.

Exact and numerical solutions at time points 40, 70, 100 for different values of N are given in Figure 1.



a ($N = 100$) ***b*** ($N = 150$) ***c*** ($N = 200$) ***d*** ($N = 250$)

Fig. 1. Comparison of numerical (dashed line) and exact (solid line) solutions at various times for the flooding problem.

5.2. Rainfall

In this example, we model the constant flow rainfall through the boundary function $f(t) = 0.044\,(1 - e^{-3t})$, resulting in a constant zero value at $t = 0$ and reaches the value 0.044 at saturation. We set $D_0 = 1400$ and $K_0 = D_0/1000$ the soil parameters. We set homogeneous Dirichlet condition at the bottom surface $g(t) = 0$ and initial soil water content $\theta_0 = 0.025$. The final time is chosen as $T = 1$ and length $L = 100$.

The analytical solution for this problem is given by [6, Equation (35)]. In Figure 2 we compare it with the numerical solution at $t = 1/15$, $1/10$ and $1/8$ for different values of N.

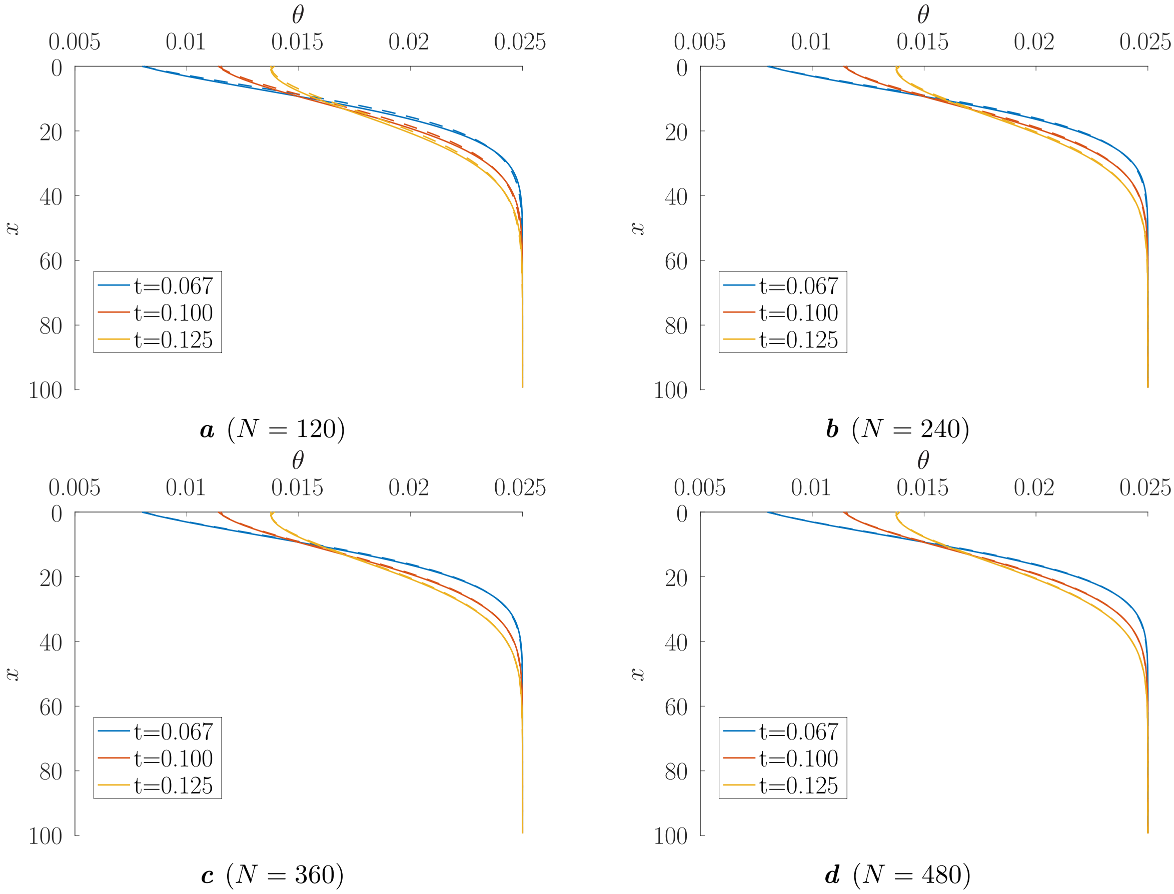


Fig. 2. Comparison of numerical (dashed line) and exact (solid line) solutions at various times for the rainfall problem.

6. Conclusions

We examined the problem of vertical infiltration in a bounded domain with length L. The mathematical model is described by a diffusion PDE together with appropriate initial and boundary conditions referring to either flooding or rainfall at the soil surface (to be placed at $x = 0$). We applied the simple and efficient two-step method, that involves the semi-discretization with respect to time using the Rothe's method and the FSM to solve the problem numerically. This is the first time, to our knowledge, that this method is used to approximate the water content in a bounded profile and it can be seen as an initial step to examine in the future the two-dimensional problem.

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Чисельне розв'язування задачі вертикальної інфільтрації в обмежених профілях

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У цій роботі розглядається чисельне розв'язування задачі одновимірної інфільтрації в обмежених профілях. Передбачається, що ґрунт має постійну водопроникність і лінійну залежність між гідропровідністю та вмістом води. Тоді задача вертикальної інфільтрації моделюється як початково-крайова задача для рівняння дифузії. Для чисельного розв'язування розглянутої задачі ми поєднуємо метод скінченних різниць по часовій змінній з методом фундаментальних послідовностей по просторовій змінній. Наведено результати чисельних експериментів для випадків затоплення та дощів, що підтверджують ефективність запропонованого алгоритму.

Ключові слова: *рівняння адвекції–дисперсії; метод Роте; метод фундаментальних послідовностей.*