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A fuzzy numerical solution of a one-dimensional steady-state heat conduction problem with a constant gradient

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Fuzzy differential equations have been gaining popularity in recent years. Traditional heat transfer models often rely on precise input parameters; however, real-world scenarios frequently involve uncertainty and imprecision. With advancements in mathematical modeling, the heat transfer increasingly used to address real-world problems. This paper presents a one-dimensional steady-state fuzzy heat transfer problem. To solve this problem, the fuzzy Runge–Kutta Cash–Karp of the fourth–order method is employed, demonstrating its effectiveness. The results are then compared to analytical solutions, revealing that the approximate solutions closely align with the analytical ones.

Keywords: *fuzzy Runge–Kutta Cash–Karp; steady-state heat conduction problem; fuzzy differential equation.*

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1. Introduction

The theory of fuzzy sets, introduced by Zadeh, addresses ambiguity and uncertainty within mathematical models, facilitating a better understanding of real-world phenomena. Differential equations, a type of mathematical model, are commonly used to represent various real-world scenarios. By incorporating fuzzy quantities in place of uncertain and imprecise classical quantities, fuzzy differential equations (FDEs) are formed [1]. Recently, the application of FDEs has grown significantly, finding utility in diverse fields such as mathematical physics, engineering, and medicine.

Numerous studies have explored fuzzy heat transfer and its applications. To enhance understanding, several of these studies have been reviewed. Jameel, Anakira, Alomari, Hashim, and Momani [2] introduced the Optimal Homotopy Asymptotic Method (OHAM) to approximate the solution of the Fuzzy Heat Equation (FHE). This method was chosen for its ability to compute the solution as an infinite series. Since this was the first time the authors applied OHAM to solve the FHE, their findings demonstrate that it converges to an analytical solution with reduced computational effort. Zureigat and Ismail [3] explored the use of the Forward Time Centered Space (FTCS) method to solve the FHE, analyzing the results with two different types of fuzzifications. To address the problem, they proposed two defuzzification approaches. Their findings showed that the second defuzzification method produced a smaller fuzzy number area compared to the first.

Tapaswini, Chakraverty, and Nieto [4] applied the Galerkin method to address heat transfer in an aluminum pin–fin from a wall surface with constant temperature. Relevant parameters were identified, and the results obtained using the proposed method were compared with those of Bede [5] and Seshu [6] for uncertain temperatures, considering quadratic, cubic, and quartic solutions. The findings for the test problems were found to be very close to the analytical solutions, demonstrating that the Galerkin method is capable of handling nth–order FDEs. In 2020, Mahardika and Haryani [7] investigated the

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time required for different tested metals to reach equilibrium temperature. Since each metal has a unique thermal conductivity, the time to reach equilibrium varies accordingly. The results indicate that metals with higher thermal conductivity reach equilibrium faster.

Imprecise measurements, fluctuating boundary conditions, or intrinsic material heterogeneity are some of the common sources of uncertainty in real-world heat transfer problems. For example, microstructural irregularities may cause a material's thermal conductivity to slightly change between areas, while ambient noise and inaccurate sensors may skew temperature measurements. Due to their inability to fully capture these uncertainties, traditional deterministic models may produce forecasts that are less accurate or inaccurate. In order to account for these uncertainties, fuzzy modeling offers a versatile and reliable framework, enabling a more accurate and insightful examination of heat conduction processes.

Therefore, in this study, a classical topic in heat transfer theory which is steady-state heat conduction in a one-dimensional solid rod is investigated. An uniform rod with constant thermal characteristics is used in the physical context, and heat moves along its length as a result of a set boundary temperature. To make analysis easier, it is assumed that heat transfer only happens by conduction and that the rod does not generate any heat inside. The objective of this study is to find out how the temperature is distributed along the rod under uncertainty condition. Fuzzy mathematics is used to handle such uncertainty in order to improve the model's realism and applicability, particularly when the exact temperature is ambiguous or inaccurate.

The paper starts by introducing the concept of one-dimensional steady-state heat conduction problem, followed by a numerical example where the Fuzzy Runge–Kutta Cash–Karp of the fourth–order method (FRKCK4) is applied to solve a one-dimensional steady-state heat conduction problem. The results are presented in both tables and figures, with the conclusions provided at the end of the paper.

2. The proposed method

This section presents the one-dimensional steady-state heat conduction problem. The model used in this study assumes a steady-state temperature gradient and is a simplified example of heat conduction. The purpose of this assumption is to make the analysis easier and to provide a basis for using the suggested fuzzy numerical method. It is important to note that this model does not represent the general form of heat conduction equations, such as the classical second-order partial differential equation commonly employed in systems that are variation. Instead, this strategy is used to emphasise how well the technique can manage uncertainty in physical parameters.

Considering the Fourier law of heat conduction with the temperature on the left surface being t_0 and the right surface being t_L, let the total amount of heat flow, Q be as follows [8]:

$$Q \propto \frac{A(t_0 - t_L)\tau}{d}. \tag{1}$$

The sign '$\propto$' is replaced by the sign '$=$' and a constant k, Eq. (1) is written as:

$$Q = k\frac{A(t_0 - t_L)\tau}{d}, \tag{2}$$

where k is the thermal conductivity coefficient, A is the cross–sectional area, τ is the time allowing heat flow, and d is the length of the metal rod.

In order to simplify Eq. (2), a steady-state heat conduction regime is assumed, in which the temperature at each point along the rod remains constant over time. With this assumption, the time-dependent (τ) of the basic heat conduction equation is removed. The heat flux, q is introduced, which represents the sense of the intensity of heat conduction. Using the heat flux, $q = \frac{Q}{A\tau}$, Eq. (2) becomes

$$q = k\frac{(t_0 - t_L)}{d}, \tag{3}$$

and the whole attention focuses on spatial variations in temperature, resulting in the formulation of Eq. (4):

$$q = k\frac{t(x) - t(x + \Delta x)}{\Delta x} = -k\frac{t(x + \Delta x) - t(x)}{\Delta x}. \tag{4}$$

The negative sign indicates that t decreases as x increases. The idea of a derivative as the limit of a difference quotient is used to transform the discrete form in Eq. (4) to the continuous formulation in Eq. (5). The derivative of temperature with respect to position is the difference quotient as the spatial interval $\Delta x \to 0$:

$$q(x) = \lim_{\Delta x \to 0} \left[-k \frac{t(x+\Delta x) - t(x)}{\Delta x} \right] = -k\, t'(x), \tag{5}$$

where Eq. (5) represents the mathematical expression of Fourier Law of Heat Conduction in the x-direction. The formulation of one-dimensional steady-state heat conduction is:

$$\begin{cases} t'(x) = -\dfrac{Q}{kA}, \quad t_0 \leqslant x \leqslant t_L, \\ t(0) = t_0. \end{cases} \tag{6}$$

In general, inaccuracies consistently affect the measurement of the solid surface temperature at time t_0, making it difficult to obtain precise readings. By applying the fuzzy setting in Eq. (6), the one-dimensional steady-state heat conduction becomes fuzzy, allowing for more accurate modelling under uncertain conditions,

$$\begin{cases} T'(x) = -\dfrac{Q}{kA}, \quad t_0 \leqslant x \leqslant t_L, \\ T(0) = T_0. \end{cases} \tag{7}$$

Applying the theory of FDE (see Theorem 1 [9]), Eq. (7) can be expressed as:

$$\begin{cases} \underline{t}'(x,\alpha) = -\dfrac{Q}{kA}, \quad \underline{t}(0,\alpha) = \underline{t}_0, \\ \overline{t}'(x,\alpha) = -\dfrac{Q}{kA}, \quad \overline{t}(0,\alpha) = \overline{t}_0. \end{cases} \tag{8}$$

By implementing the FRKCK4 [10], Eq. (8) can be approximated as follows:

$$\begin{aligned} \underline{t}(x_{i+1},\alpha) = \underline{t}(x_i,\alpha) + \Delta x \bigg(& \frac{2825}{27648}\underline{m}_1(x_i,\alpha) + \frac{18575}{48384}\underline{m}_3(x_i,\alpha) + \frac{13525}{55296}\underline{m}_4(x_i,\alpha) \\ & + \frac{277}{14336}\underline{m}_5(x_i,\alpha) + \frac{1}{4}\underline{m}_6(x_i,\alpha) \bigg), \\ \overline{t}(x_{i+1},\alpha) = \overline{t}(x_i,\alpha) + \Delta x \bigg(& \frac{2825}{27648}\overline{m}_1(x_i,\alpha) + \frac{18575}{48384}\overline{m}_3(x_i,\alpha) + \frac{13525}{55296}\overline{m}_4(x_i,\alpha) \\ & + \frac{277}{14336}\overline{m}_5(x_i,\alpha) + \frac{1}{4}\overline{m}_6(x_i,\alpha) \bigg), \end{aligned}$$

where

$$\begin{aligned} \underline{m}_1 &= -\frac{Q}{kA}, \\ \overline{m}_1 &= -\frac{Q}{kA}, \\ \underline{m}_2 &= -\frac{Q}{kA} + \frac{1}{5}\Delta x\, \underline{m}_1, \\ \overline{m}_2 &= -\frac{Q}{kA} + \frac{1}{5}\Delta x\, \overline{m}_1, \\ \underline{m}_3 &= -\frac{Q}{kA} + \frac{3}{40}\Delta x\, \underline{m}_1 + \frac{9}{40}\Delta x\, \underline{m}_2, \\ \overline{m}_3 &= -\frac{Q}{kA} + \frac{3}{40}\Delta x\, \overline{m}_1 + \frac{9}{40}\Delta x\, \overline{m}_2, \\ \underline{m}_4 &= -\frac{Q}{kA} + \frac{3}{10}\Delta x\, \underline{m}_1 - \frac{9}{10}\Delta x\, \underline{m}_2 + \frac{6}{5}\Delta x\, \underline{m}_3, \\ \overline{m}_4 &= -\frac{Q}{kA} + \frac{3}{10}\Delta x\, \overline{m}_1 - \frac{9}{10}\Delta x\, \overline{m}_2 + \frac{6}{5}\Delta x\, \overline{m}_3, \\ \underline{m}_5 &= -\frac{Q}{kA} - \frac{11}{54}\Delta x\, \underline{m}_1 + \frac{5}{2}\Delta x\, \underline{m}_2 - \frac{70}{27}\Delta x\, \underline{m}_3 + \frac{35}{27}\Delta x\, \underline{m}_4, \end{aligned}$$

$$\overline{m}_5 = -\frac{Q}{kA} - \frac{11}{54}\Delta x\,\overline{m}_1 + \frac{5}{2}\Delta x\,\overline{m}_2 - \frac{70}{27}\Delta x\,\overline{m}_3 + \frac{35}{27}\Delta x\,\overline{m}_4,$$
$$\underline{m}_6 = -\frac{Q}{kA} + \frac{1631}{55296}\Delta x\,\underline{m}_1 + \frac{175}{512}\Delta x\,\underline{m}_2 + \frac{575}{13824}\Delta x\,\underline{m}_3 + \frac{44275}{110592}\Delta x\,\underline{m}_4 + \frac{253}{4096}\Delta x\,\underline{m}_5,$$
$$\overline{m}_6 = -\frac{Q}{kA} + \frac{1631}{55296}\Delta x\,\overline{m}_1 + \frac{175}{512}\Delta x\,\overline{m}_2 + \frac{575}{13824}\Delta x\,\overline{m}_3 + \frac{44275}{110592}\Delta x\,\overline{m}_4 + \frac{253}{4096}\Delta x\,\overline{m}_5.$$

The next section presents numerical example using the proposed method.

3. Numerical example

This section presents a numerical example to demonstrate the capability of the FRKCK4. Consider a metal rod with a cross-sectional area, $A = 1005\,\mathrm{mm}^2$ and length, $x = 0.3\,\mathrm{m}$. Its circumference is thermally insulated and one end of the rod is exposed to a heat source, $Q = 12\,\mathrm{J}$. The rod material's thermal conductivity is determined by $k = 114\,\mathrm{kW/m\cdot°C}$ [8]. Through the substitution of the provided data into Eq. (8), we obtain

$$\begin{cases} T'(x) = -\dfrac{12}{114(1005\times10^{-6})}, & 0 \leqslant x \leqslant 0.3, \\ T(0,\alpha) = (45+5\alpha, 55-5\alpha), & 0 \leqslant \alpha \leqslant 1. \end{cases} \tag{9}$$

Equation (9) is translated into the FDEs as follows:

$$\begin{cases} \underline{t}'(x) = -\dfrac{12}{114(1005\times10^{-6})}, & 0 \leqslant x \leqslant 0.3, \\ \overline{t}'(x) = -\dfrac{12}{114(1005\times10^{-6})}, & \\ \underline{t}(0,\alpha) = 45+5\alpha, & \\ \overline{t}(0,\alpha) = 55-5\alpha\,, & 0 \leqslant \alpha \leqslant 1. \end{cases} \tag{10}$$

The analytical solution is:

$$\underline{\tilde{t}}(x,\alpha) = (55-5\alpha) - \frac{12}{114(1005\times10^{-6})}t,$$
$$\tilde{\overline{t}}(x,\alpha) = (55-5\alpha) - \frac{12}{114(1005\times10^{-6})}t.$$

With the application of the FRKCK4, Eq. (10) is evaluated using step size $\Delta x = 0.003$, $N = 100$, and the solutions are corrected to 17 significant figures. Table 1 provides a comparison between the analytical and approximate solutions derived using the FRKCK4 method.

Table 1. The comparisons of analytical and approximate solutions of FRKCK4 at $x = 0.15$.

α	$\underline{\tilde{t}}(0.15,\alpha)$	$\tilde{\overline{t}}(0.15,\alpha)$	$\underline{t}(0.15,\alpha)$	$\overline{t}(0.15,\alpha)$
0	29.289080911233309	39.289080911233313	29.289080911233327	39.289080911233327
0.1	29.789080911233309	38.789080911233313	29.789080911233327	38.789080911233327
0.2	30.289080911233309	38.289080911233313	30.289080911233327	38.289080911233327
0.3	30.789080911233309	37.789080911233313	30.789080911233327	37.789080911233327
0.4	31.289080911233309	37.289080911233313	31.289080911233327	37.289080911233327
0.5	31.789080911233309	36.789080911233313	31.789080911233327	36.789080911233327
0.6	32.289080911233313	36.289080911233313	32.289080911233327	36.289080911233327
0.7	32.789080911233313	35.789080911233313	32.789080911233327	35.789080911233327
0.8	33.289080911233313	35.289080911233313	33.289080911233327	35.289080911233327
0.9	33.789080911233313	34.789080911233313	33.789080911233327	34.789080911233327
1.0	34.289080911233313	34.289080911233313	34.289080911233327	34.289080911233327

Table 1 compares the analytical and numerical solutions of FRKCK4 at $x = 0.15$. The table showed that the FRKCK4 and analytical solutions are comparable, suggesting that FRKCK4 can handle the one-dimensional steady-state heat conduction problem. Figure 1 illustrates the analytical and approximate solutions of FRKCK4. As seen in the figure, the temperature distribution decreases as the length of the metal rod increases.

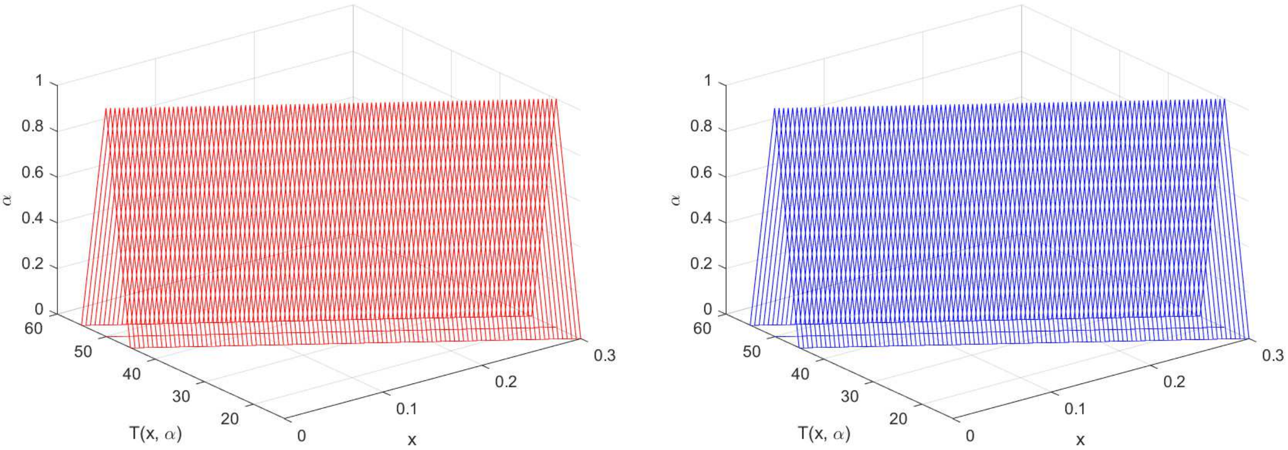

Fig. 1. The analytical solution (left) and approximate solution of FRKCK4 (right).

4. Conclusion

The FRKCK4 method was proposed to demonstrate its effectiveness in solving one-dimensional steady-state heat conduction problem. The results show that FRKCK4 effectively handles uncertainties in temperature variations by providing approximate solutions that closely align with analytical ones. By using fuzzy variables, it models vague and imprecise data, providing a range of possible outcomes rather than exact values. This leads to more accurate predictions and robust designs, especially in complex systems with uncertain conditions. Therefore, by applying FRKCK4, it can estimate the one-dimensional steady-state heat conduction rates in complex systems where traditional methods struggle due to uncertain boundary conditions. As a result, this method holds potential for extension to higher-order applications in the future.

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Нечітке чисельне розв'язання одновимірної стаціонарної задачі теплопровідності з постійним градієнтом

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Нечіткі диференціальні рівняння набувають популярності в останні роки. Традиційні моделі теплопередачі часто покладаються на точні вхідні параметри; однак сценарії реального світу часто включають невизначеність та неточність. З розвитком математичного моделювання теплопередача все частіше використовується для вирішення реальних проблем. У цій статті представлена одновимірна стаціонарна задача нечіткої теплопередачі. Для її розв'язання застосовано нечіткий метод Рунге–Кутти Кеша–Карпа четвертого порядку, що демонструє його ефективність. Отримані результати потім порівнюються з аналітичними розв'язками, показуючи, що наближені розв'язки тісно узгоджуються з аналітичними.

Ключові слова: *нечіткий метод Рунге–Кутта Кеша–Карпа; задача стаціонарної теплопровідності; нечітке диференціальне рівняння.*