MATHEMATICAL MODELING AND COMPUTING, Vol. 12, No. 2, pp. 621–627 (2025)

Generalization of certain centrality measures for social networks analysis

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(Received 19 May 2024; Revised 17 June 2025; Accepted 19 June 2025)

No doubt, the majority of people have personal accounts on several social networks, which allow them to exchange information with their family and friends and to follow their interests. A social network can be defined by a set of social entities and the relationships that these social entities have with each other directly, or indirectly through the length of the links between them. For a better understanding the interest of the social communication between people, which is the main key for the process of social network analysis, we will use indicators that are supposed to determine the notion of importance in a network by identifying the most significant entities, called Centrality measures. The idea of this paper is to generalize these measures and rewrite them in a new way in order to analyze the importance of social networks through the number of their levels.

Keywords: *complex network; centrality measures; social network analysis; degree centrality; closeness centrality; betweeness centrality.*

2010 MSC: 68R10, 05C82, 91D30 **DOI:** 10.23939/mmc2025.02.621

1. Introduction

Although the concept of "complex network" currently refers to a wide variety of objects and phenomena, due to the development of several complex systems allowing the connection between components (atoms, individuals, routers). This has motivated researchers to understand and analyze the characteristics of a complex network and its properties [1, 2]. A complex network is a network with topological characteristics that do not appear in simple networks. The word "complex network" refers to the fact that it is impossible to predict the behavior of these systems from their components. Complex networks can be used to model many types of relationships and processes in physical, biological, social and information systems [2]. The internet, which can be represented by complex network, is one of the several complex systems based on interactions between a large number of elements. It represents a network of computers and routers connected to each other by physical links [2]. Indeed, social networks could be represented as complex networks by considering the accounts as vertices and the relationship between them as edges [2–4].

The notion of social networks was founded to discover if there is a relationship between the importance of a social network and the pattern of social links in which actors are embedded. Social network analysis seeks to understand the structure of networks and their entities using two main axes: the actors and the relationships between them [4, 5]. Many kinds of theories have been developed to examine and explain the properties of interactions between the different components of a social network to define the importance of each component within a network, such as centrality measures [2, 5].

In the analysis of social networks, the idea of centrality measures appeared to understand the social communication between humans starting first with the characterization of communication in small groups of people to know if there is a relationship between structural centrality and influence in the processes of exchange and sharing within the group [3]. Centrality gathers several indicators that are used to determine the role of each entity in a social network, according to its relations (direct and

indirect) and its position. In this sense, there are two main categories of centrality measures, those related to proximity such as Degree Centrality and Closeness Centrality, and those based on being an intermediate node between two other nodes such as Betweenness Centrality [3,6].

In the rest of this article, we will define the different terms used in our research, after that, we will define new generalized formulas of these measures according to the level within a network in function of $d_G^u(k)$ in order to determine the interest of centrality measures in the analysis of social networks and understand the concept of social communication between the entities of a social network by level.

2. Preliminary notes

The study of centrality is used to define the role of an individual within a social network. A social network can be represented as a simple connected, and planar undirected graph G, consisting of a set of V individuals (vertices) and a set of E edges (lines) connecting pairs of individuals. The order of graph G, denoted by n, is its number of vertices and the size of graph G is its number of edges denoted by m. The distance between two individuals u and v in graph (network) G, denoted by $d(u,v)$ represents the number of edges of the shortest path connecting these vertices in G. The degree $\deg(u)$ of the vertex u which is the number of edges incident to u. The diameter of a graph G, denoted by $D(G)$, represents the longest distance between two vertices (individuals) of this graph [2,3]. Another indicator that plays an important role in our research is the $d_G^u(k)$. It represents the number of vertices pairs of graph G that are at distance k from u [2,3]. This parameter will help us to generalize the centrality measures of a social network in order to understand the changes in its structure.

Topological indices are indicators similar to centrality measures used to analyze social networks such as the Wiener index $W(G)$ which is an index based on distance. It represents the sum of distances between all vertices of a graph G defined by the following formula: $W(G) = \frac{1}{2}\sum_{u\in V(G)} w(u,G)$ with $w(v,G) = \sum_{u\in V(G)} d(u,v)$ [7]. The degree distance index $DD(G)$ is another example of topological indices based on degree and distance defined as follows: $DD(G) = 4m(n-1)+\sum_{u\in V(G)}\sum_{k=1}^{D(G)} \deg(u)(k-2)d_G^u(k)$ [2].

2.1. Classic centrality measures

2.1.1. Classic degree centrality

In social network analysis, the classic degree centrality is an indicator used to measure the total of direct relationships of a node with others. For a simple connected and planar graph G, the vertex's degree centrality defined by the following formula [5,6]:

$$C_d(u) = \frac{\deg(u)}{n-1}.$$

The degree centrality of a social network is the sum of the degree centralities of each node, it represented as follows:

$$C_d(G) = \frac{\sum_{u\in V(G)}\left[C_d(u_{\max}) - C_d(u)\right]}{(n-1)(n-2)}, \tag{1}$$

where $C_d(u_{\max})$ is the maximum degree centrality of any entity within the social network.

2.1.2. Classic closeness centrality

In social network analysis, the classic closeness centrality measures each node according to its "closeness" to all other nodes within a social network. Closeness centrality allows assigning a score to the entities of a network based on the sum of the shortest paths to designate the individuals who are best placed and who influence more on the entire network. For a simple connected and planar graph G, the vertex's closeness centrality defined by the following formula [6,8,9]:

$$C_c(u) = \frac{1}{\sum_{v\in V(G)} d(u,v)}.$$

The closeness centrality of a social network represented by the following formula:

$$C_c(G) = (n-1)\sum\nolimits_{u\in V(G)} C_c(u). \tag{2}$$

2.1.3. Classic betweenness centrality

In social network analysis, the classic betweenness centrality is an indicator that is used to identify the entities influencing the flow of communication within a social network by calculating the sum of the short paths between two vertices v and w through the vertex u. For a simple connected and planar graph G, the vertex's betweenness centrality defined by the following formula [8–10]:

$$C_b(u) = \sum_{v \neq w \neq u \in V(G)} \frac{g_{v,w}(u)}{g_{v,w}},$$

where $g_{v,w}$ represent the total number of shortest paths from node v to node w and $g_{v,w}(u)$ represent the number of such paths that pass through u.

The Betweenness centrality of a social network G represented by the following formula [11]:

$$C_b(G) = \frac{1}{n} C_b(u). \tag{3}$$

2.2. Centrality measures in social communication

The methodology of social network analysis are used to identify and analyze the interactions between individuals in a social network. This approach has attracted researchers to dig into the field of sociology to understand human behavior, and to study the social relationships between the different components of a social network and their consequences.

As already mentioned, in the context of social communication, centrality was first applied to human communication in small groups of people to determine the relationship between the position of individuals and their influences within a group [3, 10, 11]. In the article [12], the authors represent a general review of social network analysis by discussing the different problems recognized in this analysis such as the social communication. Another example of the use of centrality measures could be mentioned, the authors of the article [13] discussed the general concept of centrality measures and presented some applications of these indicators in social network analysis. Researchers have proposed indicators known as measures of centrality to quantify the notion of importance in terms of degree, closeness and betweenness. Knowing the important nodes in the network allows us to adopt strategies to protect them, as they play an essential role in communication.

3. Main results

In order to define the importance of each social network, we must define the importance of each component of this social network. In this sense, we tried in the article [3] to characterize the importance of each entity within the social network by treating each person in a group (level), we proposed a method of analysis of social networks based on the level of importance by generalizing the three measures of centrality according to $d_G^u(k)$. In this section, we will improve our results by trying to calculate the centrality measures of social networks by level to discuss the importance of a social network by group and define if there is a relation between the influence of each individual and the importance of a social network.

3.1. Degree centrality

Theorem 1. *Let G be a simple connected planar graph of diameter $D(G)$ with n vertices, m edges, then the Degree Centrality by level of a social network G, is defined as:*

$$C_d^k(G) = \frac{\sum_{u \in V(G)} \left[\left| -\sum_{k=2}^{D(G)} d_G^{u_{\max}}(k) \right| - \left| -\sum_{k=2}^{D(G)} d_G^u(k) \right| \right]}{(n-2)}. \tag{4}$$

Proof. Let G be a simple connected planar graph of diameter $D(G)$ with n vertices, m edges, then the Degree Centrality by level of a vertex u, is defined as [3]:

$$C_d^k(u) = \left| -\frac{1}{n-1} \sum_{k=2}^{D(G)} d_G^u(k) \right|. \tag{5}$$

Using (1) and (5), we have:

$$\begin{aligned}
C_b(G) &= \frac{\sum_{u\in V(G)}\left[C_d(u_{max}) - C_d(u)\right]}{(n-1)(n-2)} \\
&= \frac{\sum_{u\in V(G)}\left[\left|-\frac{1}{n-1}\sum_{k=2}^{D(G)} d_G^{u_{\max}}(k)\right| - \left|-\frac{1}{n-1}\sum_{k=2}^{D(G)} d_G^{u}(k)\right|\right]}{(n-1)(n-2)} \\
&= \frac{\sum_{u\in V(G)}\frac{1}{n-1}\left[\left|-\sum_{k=2}^{D(G)} d_G^{u_{max}}(k)\right| - \left|-\sum_{k=2}^{D(G)} d_G^{u}(k)\right|\right]}{(n-1)(n-2)} \\
&= \frac{\frac{1}{n-1}\sum_{u\in V(G)}\left[\left|-\sum_{k=2}^{D(G)} d_G^{u_{max}}(k)\right| - \left|-\sum_{k=2}^{D(G)} d_G^{u}(k)\right|\right]}{(n-1)(n-2)}, \\
C_d^k(G) &= \frac{\sum_{u\in V(G)}\left[\left|-\sum_{k=2}^{D(G)} d_G^{u_{\max}}(k)\right| - \left|-\sum_{k=2}^{D(G)} d_G^{u}(k)\right|\right]}{(n-2)}.
\end{aligned}$$

■

3.2. Closeness centrality

Theorem 2. *Let G be a simple connected planar graph of diameter $D(G)$ with n vertices, m edges, then the Closeness Centrality by level of a social network G, is defined as:*

$$C_c^k(G) = (n-1)\sum_{u\in V(G)}\frac{1}{\sum_{k=1}^{D(G)} k d_G^u(k)}. \tag{6}$$

Proof. Let G be a simple connected planar graph of diameter $D(G)$ with n vertices, m edges, then the Closeness Centrality by level of a vertex u, is defined as [3]:

$$C_c^k(u) = \frac{1}{\sum_{k=1}^{D(G)} k d_G^u(k)} \tag{7}$$

Using (2) and (7), we have:

$$\begin{aligned}
C_c(G) &= (n-1)\sum_{u\in V(G)} C_c(u), \\
C_c^k(G) &= (n-1)\sum_{u\in V(G)}\frac{1}{\sum_{k=1}^{D(G)} k d_G^u(k)}.
\end{aligned}$$

■

3.3. Betweenness centrality

Theorem 3. *Let G be a simple connected planar graph of diameter $D(G)$ with n vertices, m edges, then the Betweenness Centrality by level of a social network G, is defined as:*

$$C_b^k(G) = \frac{1}{n}\sum_{u\in V(G)}\sum_{k=1}^{D(G)} d_G^u(k) - \frac{n-1}{2}. \tag{8}$$

Proof. Let G be a simple connected planar graph of diameter $D(G)$ with n vertices, m edges, then the Betweeness Centrality by level of a vertex u, is defined as [3]:

$$C_b^k(u) = \sum_{k=1}^{D(G)} d_G^u(k) - \frac{n-1}{2}. \tag{9}$$

Using (3) and (9), we have:

$$C_b(G) = \frac{1}{n}C_c(u) = \frac{1}{n}\left[\sum_{k=1}^{D(G)} d_G^u(k) - \frac{n-1}{2}\right] = \frac{1}{n}\sum_{u\in V(G)}\left[\sum_{k=1}^{D(G)} d_G^u(k) - \frac{n-1}{2}\right]$$

$$= \frac{1}{n}\sum_{u\in V(G)}\sum_{k=1}^{D(G)} d_G^u(k) - \frac{1}{n}\sum_{u\in V(G)}\frac{n-1}{2} = \frac{1}{n}\sum_{u\in V(G)}\sum_{k=1}^{D(G)} d_G^u(k) - \frac{1}{n}\frac{n(n-1)}{2}.$$

■

3.4. Centrality measures in social communication by levels

The importance of a social network resides in the importance of its components in terms of direct and indirect relationships as well as the type of information shared. In other words, in social networks, some entities play a very important role in the social communication between individuals while others have no influence on the network. In this sense, researchers have noticed that some actors play a more "important" role than others, for this reason, they have proposed the centrality measures to quantify the notion of the importance of a network in terms of the importance of its components. The centrality of a network can be used to define to most influenced entities in social networks (e.g., a group of people, a community, etc.). In our case, we are interested to capture the notion of the importance of a social network by identifying its central components.

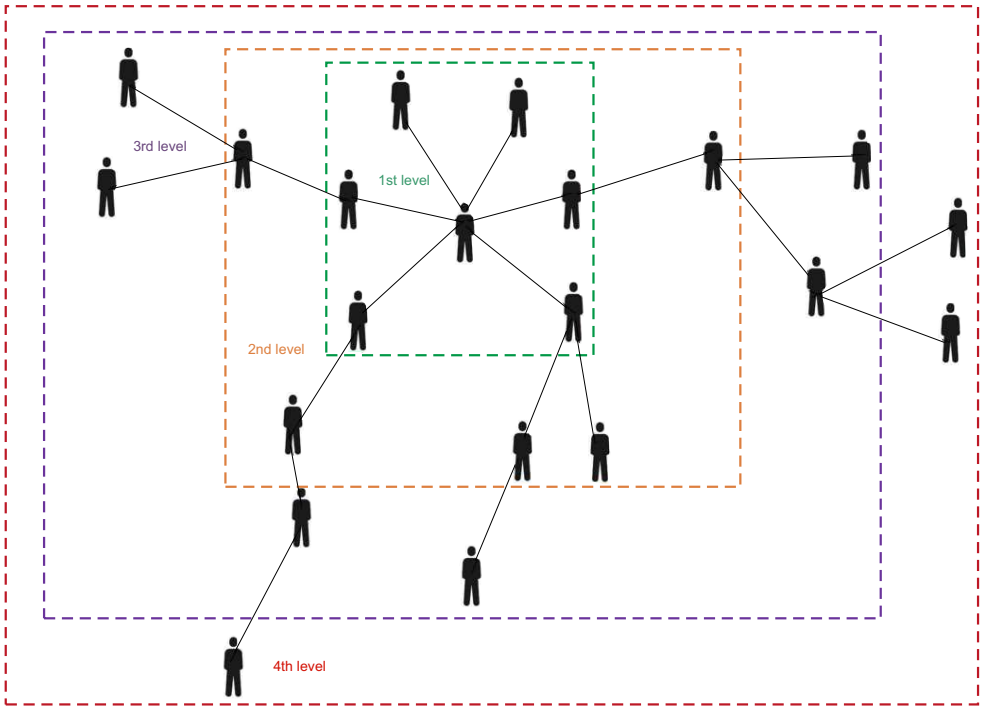

Fig. 1. The social network G composed by N individuals.

Understanding the flow of sharing and exchanging information between individuals in a social network constitutes the main objective of the study and analysis of social networks. The measures of centrality allow to indicate the importance of networks (graphs) in terms of individuals. These indicators measure the importance of a user in terms of direct and indirect relations, in other words, the number of users who are far from the individual u by 1 represents the first level of the relationship of this individual with the other components of a social network noted by $d_G^u(1)$. The second level represents the number of users far from an entity u by a distance equal to 2 ($d_G^u(2)$), which means the importance of the neighboring entities of this individual u according to the number of their direct relations (the degree of the neighboring entities). Our method will come to its end when we reach the last level ($D(G)$).

Our main objective is to determine the relationship between the centrality measures of a social network and the number of its levels using the $d_G^u(k)$. This will allow us to classify the relationships between individuals within a social network according to its levels and to understand the process of information diffusion between the components. Based on this method, we can define the individuals who influence more within a social network as well as the relationship between its importance and the importance of its different components.

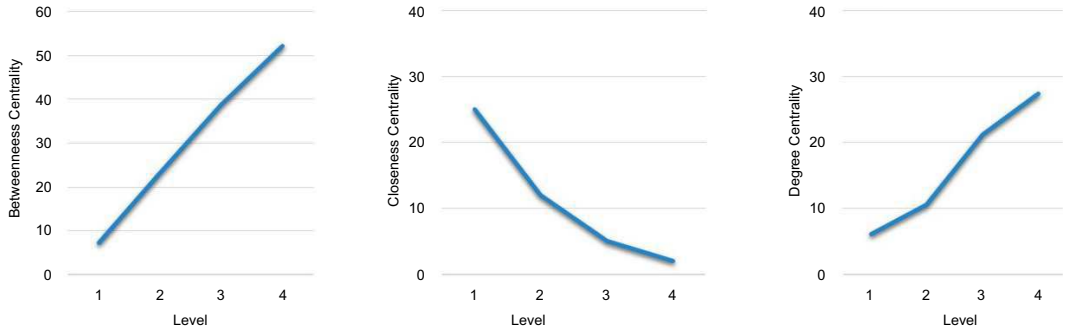

Fig. 2. The variance of the importance of a social network according to its levels.

The graph above shows us the variance of the importance of a social network ($C_d^k(G)$, $C_c^k(G)$, $C_b^k(G)$) in terms of the number of levels that compose this network and the number of individuals. To see this difference, we tried to calculate the three measures of centrality of the social network (Figure 1) based on the new formulas proposed in function of $d_G^u(k)$ to understand the behavior of each entity that composes this network. According to the obtained results, we can notice that the importance of a social network depends on the importance of its entities, in the first case, we can notice that the importance of the network (Figure 1) decreases when the number of levels increases, it is about the degree centrality and closeness centrality ($C_d^k(G)$, $C_c^k(G)$), on the other hand, the importance of the network increases when the number of levels increases in the case of the measure of betwenness centrality ($C_b^k(G)$).

4. Conclusion

The importance of an individual within a social network depends on the number of direct and indirect friendship relationships with other individuals that you can reach from it. Social network analysis allows to measure the importance of each person as a function of the social relationships within a social network. In this context, centrality measures have been proposed to provide a ranking that identifies the most important nodes in social network analysis.

Our main research work has to present an overview about the most centrality measures used by generalizing these indicators in function of $d_G^u(k)$ to understand the concept of social communication between the entities of a social network by level and define if there is a relation between the influence of each individual and the importance of a social network.

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Генералізація певних мір центральності для аналізу соціальних мереж

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Безумовно, більшість людей мають особисті акаунти в кількох соціальних мережах, що дозволяє їм обмінюватися інформацією з родиною та друзями та стежити за своїми інтересами. Соціальна мережа може бути визначена як набір соціальних сутностей та зв'язків, які ці соціальні сутності мають одна з одною безпосередньо або опосередковано через довжину зв'язків між ними. Для кращого розуміння інтересу соціальної комунікації між людьми, що є основним ключем для процесу аналізу соціальних мереж, ми будемо використовувати показники, які повинні визначати поняття важливості в мережі, виявляючи найбільш значущі сутності, які називаються мірами центральності. Ідея цієї роботи полягає в генералізації цих мір і переписуванні їх у новий спосіб для аналізу важливості соціальних мереж через кількість їх рівнів.

Ключові слова: *складна мережа; міри центральності; аналіз соціальних мереж; ступенева центральність; близькість центральності; центральність через посередництво.*