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MATHEMATICAL MODEL OF THE BOUNDARY LAYER OF AIRFLOW OVER A FLAT SURFACE

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Abstract. In fluid and gas mechanics, a subject that has been the focus of considerable scholarly attention is the modeling of flow phenomena on streamlined surfaces. These surfaces, which are typically installed in a parallel orientation to the direction of the free stream, play a pivotal role in the study of fluid dynamics. The boundary layer theory represents a pivotal branch of fluid dynamics, given the airflow plate is characterized by a high Reynolds number of airflow velocity entering the plate's plane surface contexts. The complexity of the problem is due to the nonlinearity and multidimensional nature of the governing equations. This study proposes a straightforward numerical methodology that can address a range of nonlinear problems in surface flow mechanics, particularly in the context of near-wall boundary layers of a planar nature. The paper considers the process of air motion as an isotropic Newtonian medium on the surface as an isotropic Newtonian medium layer. The resulting differential equation is expressed in dimensionless quantities and then solved numerically using the Runge-Kutta method. The velocity distribution in the boundary layer on a flat airflow plate is obtained. As the airflow velocity entering the plate surface increases, so too do the tangential stresses. The nature of the change in tangential stresses is linear in the initial coordinate, corresponding to the onset of airflow entering the plate surface, with two transition points identified at Mach number M = 1 and M = 3. It is evident that along the entire length of the surface of the flat plate, the nature of the change in tangential stresses is not a linear dependence. Consequently, with an increase in the distance from the end face of the plate from 1 mm to 100 mm (a 100-fold increase), the tangential stresses decrease by 10 times within a given length interval. Furthermore, within the length interval ranging from 0.1 m to 1.0 m, the tangential stresses undergo a reduction by a factor of 3. The presented method of modeling the distribution of velocity and tangential stresses in the boundary layer on a flat airflow surface makes it possible to calculate the force loads on the surface in the entire range of flow velocities for an incompressible medium.

Keywords: model, stress, Reynolds number, flat plate, acceleration, velocity, boundary layer, Mach number, viscosity, dimensionless number.

Introduction

In fluid and gas mechanics, a subject that has been the focus of considerable scholarly attention is modeling flow phenomena on streamlined surfaces. These surfaces, which are typically installed in a parallel orientation to the direction of the free stream, have been the subject of numerous studies. Despite the extensive research and notable findings in addressing this problem for the laminar flow regime [1–4], several aspects of such flows have received insufficient attention. One such example is the interaction mechanism of the boundary flow in the near-wall layer directly on the surface itself, which affects the force characteristics of the movement process of both the boundary layer and the flow body. This is important for solving various technical and gas-dynamic problems, yet it has been given relatively little attention [5–9].

An effective approach to solving the problem of external flow with a high Reynolds number is known as the boundary layer analysis technique, which was first developed by Prandtl in 1904.

One of his students, Blasius, in 1908 presented a technique for transforming the well-known problem of laminar boundary layer flow over a flat plate into an ordinary differential equation (ODE). The Blasius equation is of great importance in many engineering applications, as it provides very good approximations for the boundary layer thickness and the total drag force in laminar external flows [2].

Blasius himself [3] proposed an approximate solution with an infinite series, which converges only for small values, reaching an estimate for the angle of incidence $''(0)$ [4], with a relative error of 8.6%, using the perturbation method and Padè approximated, [5] giving a simple approach called the iterative perturbation method, and obtaining $''(0)$ with a relative error of 0.73%, which can be considered a very good result, considering the simplicity of the calculations. Using the variational iteration method, [6 - 8] gives valid solutions for the entire domain. A solution with numerical transformations was presented in [9], while in [10] an estimation method is used to determine the Taylor coefficients. The kernel reproduction method has been successfully applied by Kernel [11]. Among all the numerical solutions, [12] is the most famous for its extraordinary accuracy and is generally considered the exact result for comparison. Some other solution methods applied to the Blasius equation are the Sinc-collocation method [13], the homotopy analysis method [14, 15], the Laguerre-collocation method [16], the homotopy perturbation method [15], the parameter iteration method [17], the differential transformation method [18, 19], the Adomian's decomposition method [20, 21] and the modified rational Legendre tau method [22].

First, the solution of the Blasius series (the solution with the perturbation technique) is considered, and the range of the series is extended by the Padè approximation [23]. Second, the Galerkin-based weighted residual method [24] is applied with two trial functions. Also, for the analytical solution of the equations of the laminar boundary layer over a flat plate, the homotopic perturbation method (HPM) is used to solve the well-known nonlinear differential equation of Blasius [25]. This method is close to the exact solution of the equation. To solve the Blasius equation, a nonlinear differential equation of the third order, a feedforward neural network was used [26], where the Blasius equation was solved without reducing it to a system of first-order equations.

Similarly, for the classical Blasius problem, a noniterative transformation method based on scale invariance properties has been used [27]. This method allows numerically solving the boundary value problem by solving the initial problem and then scaling the resulting numerical solution.

Another way to solve the Blasius boundary layer is to expand the function $f(\eta)$ into a Taylor series with progressively shifted centers of expansion (jumping centers) [28], or by using asymptotic variation parameters [29]. In [28], the Newton-Raphson method was used to calculate the value of the unknown initial condition iteratively.

Alternatively, Jaguaribe [30] showed that an infinite number of "solutions" can be constructed that satisfy the classical equation and its three boundary layer conditions. However, the problem of the technique of reducing the boundary value problem, BVP, to the initial problem, IVP, and the technique of perturbation analysis, which is used to include a fourth boundary condition in the differential equation, are controversial and have some uncertainty. For the extended Blasius equation, taking into account its well-known classical boundary conditions, where the viscosity coefficient is assumed to be positive and depends on temperature, which arises in several important boundary layer problems in fluid dynamics, an explicit approximate solution has been obtained [31].

Problem Statement

Boundary layer theory represents a pivotal branch of fluid dynamics, given that external flows characterized by high Reynolds numbers are prevalent in natural phenomena and numerous engineering applications. The resolution of these problems typically necessitates intricate endeavors, owing to the nonlinearity and multidimensional character of the governing equations. While a sufficient number of exact solutions exist for the full Navier-Stokes equations, these are only valid for some special cases and geometries.

The objective of the present study lies in formulating a novel simple numerical method, one that may be utilized in a range of nonlinear problems pertaining to surface flow mechanics in near-wall boundary layers of planar shape.

Mathematical model of the boundary layer of flow around a flat surface

Let us consider the process of air movement on the surface of a plate in the near-wall boundary layer. The flow continuity equation can be represented as [2]

$$\frac{\partial \rho}{\partial t} + \vec{v} \cdot \mathrm{grad}\rho + \rho \cdot \mathrm{div}\vec{v} = 0, \tag{1}$$

where $\frac{\partial \rho}{\partial t}$ - is the stationary (local) process of changing the density ρ of air in time *t*; $\vec{v} \cdot \mathrm{grad}\rho$ – is the convective component, the dependence on the change in position in the reference frame.

For an incompressible medium, we have

$$\frac{\partial \rho}{\partial t} + \vec{v} \cdot \mathrm{grad}\rho = 0; \ \mathrm{div} = 0. \tag{2}$$

The momentum equation, characterizing the force according to Newton's second law, in vector form will be:

$$\rho \cdot \left(\frac{\partial \vec{v}}{\partial t} + \frac{d\vec{v}}{dt}\right) = \vec{F} + \vec{P}\,, \tag{3}$$

where $\vec{F}$ - is the air force per unit volume; $\vec{P}$ – is the surface pressure force per unit volume; $\frac{\partial \vec{v}}{\partial t}$ – is the local acceleration during unsteady air motion; $\frac{d\vec{v}}{dt}$ – is the convective acceleration during changes in position in the reference frame.

$$\frac{d\vec{v}}{dt} = \mathrm{grad}\left(\frac{\vec{v}^2}{2}\right) - \vec{v} \times \mathrm{curl}\vec{v}\,. \tag{4}$$

Consider an isotropic Newtonian medium, air. We will assume that the airflow is formed to move along the X-axis, which is co-directed with the air velocity vector $\vec{v} \approx U$. Vertical axis Y, no airflow occurs, but the dependence $U \approx f(Y)$ applies; the velocity in the direction of the X axis is variable along the height of the Y ordinate, but the velocity vector is directed along the X ordinate. Accordingly, $\vec{v} \approx V(Y) = 0$. Along the horizontal ordinate Z, perpendicular to the X ordinate, we will initially assume that there is no airflow.

The Navier-Stokes equation [2] for this problem will take the form:

momentum equation

$$\rho \cdot \left(U \cdot \frac{\partial U}{\partial X} + V \cdot \frac{\partial U}{\partial Y}\right) = -\frac{\partial p}{\partial X} + \mu \cdot \left(\frac{\partial^2 U}{\partial X^2} + \frac{\partial^2 U}{\partial Y^2}\right), \tag{5}$$

flow continuity equation

$$\frac{\partial(\rho \cdot U)}{\partial X} + \frac{\partial(\rho \cdot U)}{\partial Y} = 0, \tag{6}$$

where *U, V* - are the air flow velocities in the direction of the X and Y axes, m/s; p - pressure, in the volume of the boundary layer space, $p = \rho \cdot U_\infty^2$, Pa; ρ – air density, kg/m^3; U_∞ - air velocity at the entrance to the plate and at the boundary layer boundary, m/s; μ - dynamic viscosity of air, Pa·s.

Applying the principle of similarity, we introduce dimensionless coordinates by analogy with [2]. The thickness of the near-wall boundary layer is quantitatively estimated as

$$\delta \sim \sqrt{\frac{\nu \cdot x}{U_\infty}}, \tag{7}$$

where ν – is the kinematic viscosity of air, m^2/s.

Instead of the coordinate Y, we introduce the dimensionless coordinate η:

$$\eta = \frac{y}{\delta} = y \cdot \sqrt{\frac{U_\infty}{\nu \cdot x}} \ . \tag{8}$$

To integrate the flow continuity equation (6), we introduce the flow function, by analogy with [2]:

$$\psi = \sqrt{\nu \cdot x \cdot U_\infty} \cdot f(\eta), \tag{9}$$

where $f(\eta)$ – is a dimensionless flow function.

Then for the longitudinal velocity in the direction of the X-axis, we have

$$U = \frac{\partial \psi}{\partial y} = \frac{\partial \psi}{\partial \eta} \cdot \frac{\partial \eta}{\partial y} = \frac{\partial \psi}{\partial \eta} \cdot \sqrt{\frac{U_\infty}{\nu \cdot x}} \ . \tag{10}$$

Accordingly

$$\frac{\partial \psi}{\partial \eta} = \sqrt{\nu \cdot x \cdot U_\infty} \cdot f'(\eta) \ . \tag{11}$$

Substituting equation (11) into equation (10), we obtain

$$U = U_\infty \cdot f'(\eta) \ . \tag{12}$$

Accordingly

$$V = -\frac{\partial \psi}{\partial x} = -\frac{1}{2} f(\eta) \sqrt{\frac{U_\infty \cdot \nu}{x}} - \sqrt{\nu \cdot x \cdot U_\infty} \cdot f'(\eta) \cdot \frac{\partial \eta}{\partial x}, \tag{13}$$

$$\frac{\partial \eta}{\partial x} = -\frac{1}{2} y \sqrt{\frac{U_\infty}{\nu \cdot x^3}} . \tag{14}$$

Substitute equation (14) into (13):

$$V = \frac{1}{2} \sqrt{\frac{\nu \cdot U_\infty}{x}} \cdot \big(\eta \cdot f'(\eta) - f(\eta)\big). \tag{15}$$

Let us write down the first and second-order differentials of the obtained velocity equations:

$$\frac{\partial U}{\partial y} = U_\infty \cdot \sqrt{\frac{U_\infty}{\nu \cdot x}} \cdot f''(\eta), \tag{16}$$

$$\frac{\partial U}{\partial x} = -\frac{1}{2} \frac{U_\infty}{x} \cdot \eta \cdot f''(\eta), \tag{17}$$

$$\frac{\partial^2 U}{\partial y^2} = \frac{U_\infty^2}{\nu \cdot x} \cdot f'''(\eta) \ , \tag{18}$$

$$\frac{\partial^2 U}{\partial x^2} = \frac{3}{4} \frac{U_\infty}{x^2} \cdot \eta \cdot f'''(\eta) \ . \tag{19}$$

We will assume that the boundary conditions on the surface of the plate are such that the vertical velocity will be $V = 0$ at $y = 0$.

We assume that the velocity of the potential flow is *constant*, then

$$\frac{dP}{dy} \equiv 0\,.$$

Equation (5) taking into account (16), (17), (18), and (19), and taking into account that $\mu/\rho = \nu$, we obtain:

$$-U_\infty f' \frac{1}{2}\frac{U_\infty}{x}\eta f'' + \frac{(\eta f' - f)U_\infty}{2}\sqrt{\frac{\nu U_\infty}{x}}\sqrt{\frac{U_\infty}{\nu x}} f'' = \nu\left(\frac{3}{4}\frac{U_\infty}{x^2}\eta f''' + \frac{U_\infty^2}{\nu x} f'''\right), \tag{20}$$

having carried out the corresponding transformations and reductions of equation (20), we obtain

$$-f \cdot f'' - f'''\left(\frac{3}{2}\frac{\nu}{U_\infty x}\cdot\eta + 2\right) = 0\,, \tag{21}$$

or

$$f \cdot f'' + f'''\left(\frac{3}{2}\frac{\nu}{U_\infty x}\cdot\eta + 2\right) = 0. \tag{22}$$

Considering that

$$\frac{\nu}{U_\infty x} = Re^{-1},$$

equation (22) will take the form

$$f'''\left(\frac{3}{2}\frac{\eta}{Re} + 2\right) + f \cdot f'' = 0. \tag{23}$$

$$y := \begin{pmatrix} 0 \\ 0 \\ 0.333 \end{pmatrix}$$ initial conditions

$$D(t,y) := \begin{pmatrix} y_1 \\ y_2 \\ \frac{-1}{2}\cdot y_0 \cdot y_2 \end{pmatrix}$$

z := rkfixed(y, 0, 7, 1000, D)

z =

	0	1	2	3
0	0	0	0	0.333
1	$7\cdot10^{-3}$	$8.159\cdot10^{-6}$	$2.331\cdot10^{-3}$	0.333
2	0.014	$3.263\cdot10^{-5}$	$4.662\cdot10^{-3}$	0.333
3	0.021	$7.343\cdot10^{-5}$	$6.993\cdot10^{-3}$	0.333
4	0.028	$1.305\cdot10^{-4}$	$9.324\cdot10^{-3}$	0.333
5	0.035	$2.04\cdot10^{-4}$	0.012	0.333
6	0.042	$2.937\cdot10^{-4}$	0.014	0.333
7	0.049	$3.998\cdot10^{-4}$	0.016	0.333
8	0.056	$5.221\cdot10^{-4}$	0.019	0.333
9	0.063	$6.608\cdot10^{-4}$	0.021	0.333
10	0.07	$8.158\cdot10^{-4}$	0.023	0.333
11	0.077	$9.872\cdot10^{-4}$	0.026	0.333
12	0.084	$1.175\cdot10^{-3}$	0.028	0.333
13	0.091	$1.379\cdot10^{-3}$	0.03	0.333
14	0.098	$1.599\cdot10^{-3}$	0.033	0.333
15	0.105	$1.836\cdot10^{-3}$	0.035	...

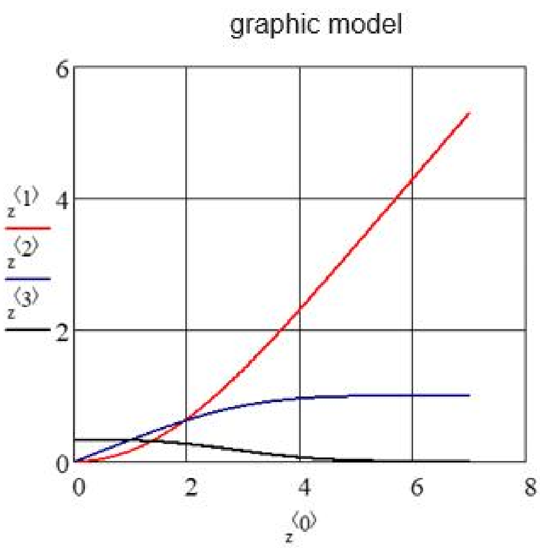

Fig. 1. An example of a numerical solution of the differential equation (23) by the Runge-Kutta method in Mathcad

The equation (23) is carried out by the numerical Runge-Kutta method in the Mathcad software environment (Fig. 1).

The initial data for the simulation are:

- dimensionless coordinate $\eta = 0 \ldots 7$;
- kinematic viscosity of air $\nu = 1{,}47 \cdot 10^{-5}$ m^2/s;
- velocity of air entering a flat plate $U_\infty = 50 \ldots 1000$ m/s;
- coordinate of the length of the flat plate $x = 0 \ldots 1$ m.

Initial conditions for numerical differentiation by the Runge-Kutta method of equation (23):

$y(0) = 0$; $y' = 0$; $y'' = 0{,}333$.

Knowing the velocity distribution in the boundary layer, we model the tangential stresses applied to the plate surface according to the formula [32]

$$\tau_x = \mu \cdot \left(\frac{\partial U}{\partial y}\right)_{y=0}. \tag{24}$$

Taking into account dependence (16), equation (24) will take the form

$$\tau_x = \mu \cdot U_\infty \cdot \sqrt{\frac{U_\infty}{\nu \cdot x}} \cdot f''(\eta) = f''(\eta)_{y\approx 0} \cdot \sqrt{\frac{{U_\infty}^3 \cdot \rho^2 \cdot \nu}{x}}. \tag{25}$$

where ρ is the air density, kg/m^3.

Results of simulation of velocity and tangential stresses in the boundary layer of a flat surface

The results of the simulation of the velocity distribution in the near-wall boundary layer of a flat plate are shown in Fig. 2. in the form of a graph, which is constructed based on the results of the numerical solution of equation (23).

The modeling of tangential stresses along the length of a flat plate from 1 mm to 1 m at air inlet speeds from 50 m/s to 1000 m/s is presented in the form of 3-D surfaces in Fig. 4. For the convenience of analyzing the modeling results, we divide the coordinate X, which coincides with the air velocity vector U_∞, into two intervals: the first interval is 0.001 - 0.1 m; the second interval is 0.1 - 1.0 m.

Simulation of tangential stresses, equation (25), Fig. 3, shows that at a velocity $U_\infty = 100$ m/s of the airflow entering at the beginning of the flat surface, the tangential stresses are $\tau_x = 4{,}7$ KPa. For a velocity $U_\infty = 330$ m/s of the airflow entering at the beginning of the flat surface, the tangential stresses are $\tau_x = 28{,}2$ KPa (Fig. 4). Similarly, at a velocity $U_\infty = 1000$ m/s the tangential stresses on the plate surface are $\tau_x = 148{,}7$ KPa (Fig. 4).

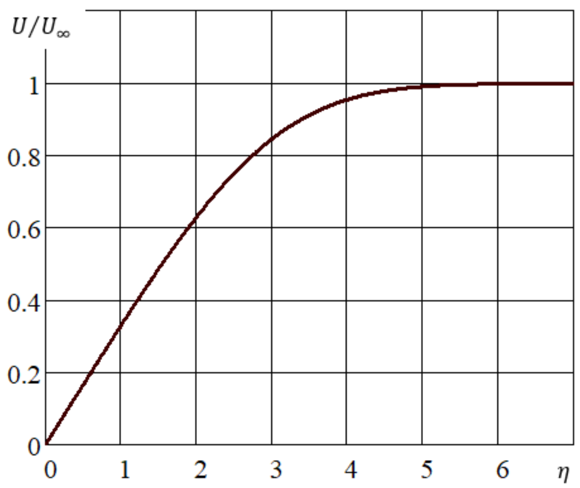

Fig. 2. Dependence of U/U_∞ on η for a laminar boundary layer of airflow around a flat plate

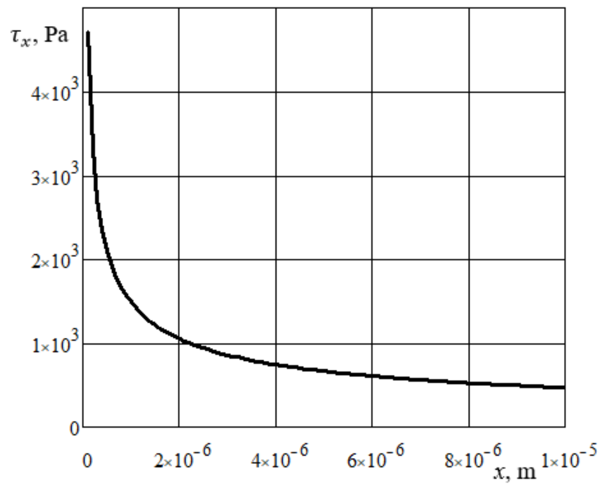

Fig. 3. Distribution of tangential stresses τ_x along the flow x on a flat surface in laminar air flow (U_∞= 100 m/s) in the boundary layer

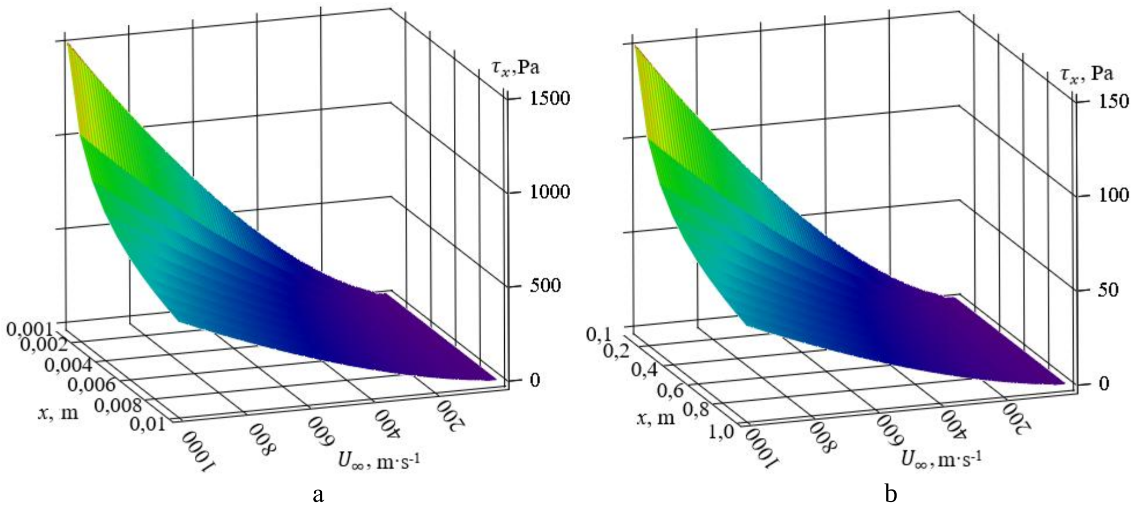

Fig. 4. Dependence of tangential stresses τ_x on the velocity U_∞ of air at the plate entrance and at the boundary layer boundary and the distance x from the plate end: a – x = 0,001…0,01 m; b – x = 0,1…1,0 m

From the end of a flat plate at a distance x = 0,001 m at an air velocity at the plate entrance and at the boundary layer boundary U_∞ = 50 m/s the tangential stresses on the plate surface are τ_x = 16,58 Pa, and at an air velocity at the plate entrance and at the boundary layer boundary U_∞ = 1000 m/s the tangential stresses on the plate surface are τ_x = 1,483 KPa.

From the end of the flat plate at a distance x = 0,1 m at an air velocity at the entrance to the plate and at the boundary layer boundary U_∞ = 50 m/s the tangential stresses on the plate surface are τ_x = 1,66 Pa, and at an air velocity at the entrance to the plate and at the boundary layer boundary U_∞ = 1000 m/s the tangential stresses on the plate surface are τ_x = 148,3 Pa.

The analysis shows that with an increase in the airflow velocity at the beginning of entering the flat surface of the plate, provided that the ordinate x ~ 0, the tangential stresses also increase (Fig. 5).

The character of change of tangential stresses at the entrance of air to a plane streamlined surface is linear (Fig. 5). However, this characteristic has two inflection points, which are characterized by a change in the proportionality coefficient. The first transition point is characterized by a transition of the velocity of air entering a flat plate, which corresponds to a Mach number equal to M = 1 (Fig. 5). The second inflection point corresponds to a Mach number M = 3.

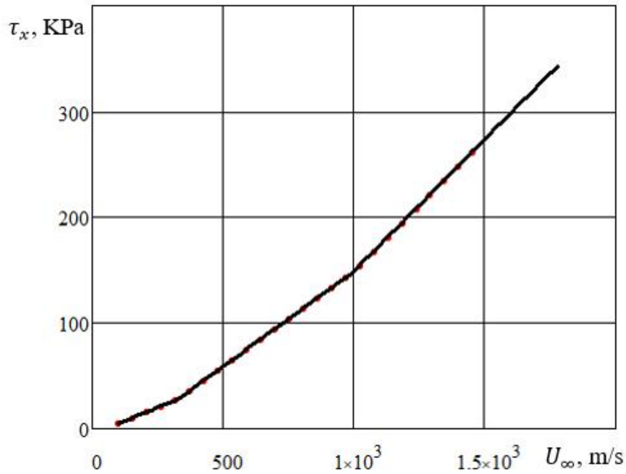

Fig. 5. Characteristic of change of tangential stresses τ_x from the velocity U_∞ of air at the entrance to the plate and at the boundary layer boundary provided that x ~ 0

Conclusions

The value of the dimensionless velocity U/U_∞, which is calculated by the numerical method, allows us to show the velocity distribution in the boundary layer on a flat airflow plate.

With increasing air flow velocity entering the plate's plane surface, tangential stresses also increase. The nature of the change in tangential stresses is linear in the initial coordinate, which corresponds to the beginning of the airflow entering the plate surface, with two transition points at Mach number M = 1 and M = 3.

Along the entire length of the surface of the flat plate, the nature of the change in tangential stresses is not a linear relationship. Thus, with an increase in the distance from the end face of the plate beginning from 1 mm to 100 mm, which is a 100-fold increase, tangential stresses decrease by 10 times over a given length interval. Over a length interval from 0.1 m to 1.0 m, tangential stresses decrease by 3 times, respectively.

The presented method of modeling the distribution of velocity and tangential stresses in the boundary layer on a flat surface of airflow makes it possible to calculate the force loads on the surface in the entire range of flow velocities for an incompressible medium.

Therefore, solving the boundary layer problem on a streamlined surface at an arbitrary Reynolds number makes it possible to implement applied problems in the field of gas-liquid flow dynamics, since external flows with high Reynolds numbers are common both in nature and in many engineering applications.

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