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Research of thermal processes in a solid bimetallic cylinder under the action of an unsteady electromagnetic field

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This study introduces a comprehensive physical and mathematical model designed to investigate thermal processes within a solid bimetallic cylindrical rod subjected to an amplitude-modulated radio pulse. The initial relations for the electrodynamics initial-boundary-value problem of the rod are presented, with the axial component of the magnetic field intensity vector identified as the key determining function. The problem is addressed by approximating the distributions of this determining function in the constituent cylinders using quadratic polynomials along the radial coordinate. The paper outlines the expression for the axial component of the magnetic field intensity vector and Joule heat in the constituent cylinders of the solid bimetallic rod under the influence of an amplitude-modulated radio pulse. Numerical analysis is then employed to investigate the time variation and radial coordinate distribution of these factors within the constituent cylinders of the bimetallic rod.

Keywords: *solid bimetallic cylinder; amplitude-modulated radio pulse; axial component of the magnetic field intensity vector; Joule heat; thermal modes.*

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1. Introduction

The study of electromagnetic processes in conductive materials is a fundamental topic explored in monograph [1], which delves into the general principles governing these phenomena. Concurrently, the application of pulsed magnetic fields in modern material processing technologies is examined in detail in [2].

Solid bimetallic cylindrical rods play a crucial role as structural components in various applications within the chemical industry, serving as cathode elements and power devices [1, 2]. These structural elements, taking the form of solid bimetallic cylindrical rods, are routinely subjected to non-stationary electromagnetic fields (EMF), in particular, amplitude-modulated radio pulses (AMRPs) during their operational processes.

Literature on heat transfer processes in solids is comprehensively detailed in [3], while [4] focuses on mathematical methodologies for solving thermal conductivity problems. Additionally, the monograph [5] discusses the theory of electric heating and its application in heating structural elements within technological processes.

The results of studies of the thermomechanical behavior of electrically conductive structural elements under induction heating by steady-state and quasi-steady-state electromagnetic fields are presented in [6].

Conversely, Ref. [7] explores the effects of AMRPs on thermoelastic energy dissipation in a hollow cylindrical sensor, highlighting gaps in the study of thermal behaviors under short-term, unstable electromagnetic field effects.

This research focuses on investigating the thermal behavior of solid bimetallic cylindrical rods (SBMCR) when subjected to electromagnetic treatment using AMRPs. Through computer analysis, this study unveils the patterns in the thermal dynamics of SBMCR, revealing their dependence on the amplitude-frequency parameters of AMRPs and the geometric characteristics of the rods.

2. Initial statements of the physical and mathematical model

Consider a solid bimetallic cylindrical rod (SBMCR) within the cylindrical coordinate system $Or\varphi z$. The axis Oz coincides with the symmetry axis of the SBMCR. The inner solid cylinder has a radius of $r = r_1$ while the outer hollow cylinder has a radius $r = r_2$. The joint surface radius of the solid and hollow component cylinders of the SBMCS is uniform and equal to $r = r_1$.

Each component cylinder ($n = 1, 2$) of the SBMCS is composed of materials that are homogeneous, isotropic, and nonferromagnetic. It is assumed that their electrophysical parameters, denoted as σ_n for electrical conductivity and μ_n for magnetic permeability, remain constant and are equal to their average values within the respective heating ranges.

For the electromagnetic treatment of SBMCR, the action of AMRP was employed. This action of AMRP is mathematically described by the following expression [7]

$$H_{0z}(t) = kH_0 \left(\exp\left(-\beta_1 t\right) - \exp\left(-\beta_2 t\right)\right) \cos\omega t. \tag{1}$$

Here H_0 is the amplitude of sinusoidal carrier electromagnetic oscillations of frequency ω; β_1, β_2 are parameters characterizing the times of the AMRP rise and fall fronts; k is normalization factor, t is time.

As a result of the action of AMRP in each n-th ($n = 1, 2$) constituent cylinder of SBMCR, eddy currents are induced. When these currents flow within the constituent cylinders of SBMCR, volumetrically distributed non-stationary Joule heat sources $Q^{(n)}$ arise.

Based on the considered physical model, to determine the Joule heat $Q^{(n)}$, it is first necessary to determine the axial component $H_z^{(n)}$ of the magnetic field intensity vector $\boldsymbol{H}^{(n)} = \left\{0; 0; H_z^{(n)}(r,t)\right\}$ in each cylinder that is different from zero.

Let us consider a plane axisymmetric problem of electrodynamics for the considered SMSC. Under such conditions, according to Maxwell's relations, the axial component $H_z^{(n)}(r,t)$ of the magnetic field intensity vector $\boldsymbol{H}^{(n)}$ is different from zero. This component $H_z^{(n)}(r,t)$ is a function of the radial variable r and time t.

To find the function $H_z^{(n)}(r,t)$ in each n-th ($n = 1, 2$) constituent cylinder of the SBMCS, the following equations

$$\frac{\partial^2 H_z^{(n)}}{\partial r^2} + \frac{1}{r}\frac{\partial H_z^{(n)}}{\partial r} - \sigma_n \mu_n \frac{\partial H_z^{(n)}}{\partial t} = 0 \tag{2}$$

were obtained on the basis of Maxwell's relations.

Equation (2) is solved under the boundary condition

$$H_z^{(2)}(r_2, t) = H_{z0}^{+}(t) \tag{3}$$

on the outer surface $r = r_2$ of the SBMCS and the condition

$$\frac{\partial H_z^{(1)}(0,t)}{\partial r} = 0 \tag{4}$$

on the cylinder axis $r = 0$. This condition corresponds to the axisymmetry of the magnetic field on the cylinder axis. On the surface $r = r_1$ of the connection of the SBMCS constituent cylinders, the conditions of perfect electromagnetic contact are observed

$$H_z^{(1)}(r_1, t) = H_z^{(2)}(r_1, t), \quad \frac{\partial H_z^{(1)}(r_1,t)}{\partial r} = k_\sigma \frac{\partial H_z^{(2)}(r_1,t)}{\partial r}. \tag{5}$$

Here $k_\sigma = \sigma_1/\sigma_2$. In the absence of AMRP at a given time $t = 0$, the initial condition is

$$H_z^{(n)}(r, 0) = 0. \tag{6}$$

By solving the initial boundary value problem (2)–(6), we obtain the expression for the component $H_z^{(n)}$ within the n-th constituent cylinder of the SBMCS. Using the found function $H_z^{(n)}$, according to Maxwell's relations, we formulate the expression of the Joule specific heat density

$$Q^{(n)} = \frac{1}{\sigma_n}\left(\frac{\partial H_z^{(n)}}{\partial r}\right)^2 \tag{7}$$

in n-th composite cylinder of the SBMCS.

3. Determination of the Joule heat under AMRP action

According to the considered physical-mathematical model, in the initial stage, we first find the solution to the initial-boundary-value problem (2)–(6). To achieve this, we approximate the distributions of the functions $H_z^{(n)}$ in each n-th component cylinder of the SBMCR with quadratic polynomials

$$H_z^{(n)}(r,t) = \sum_{i=0}^{2} a_i^{(n)}(t)\, r^i. \tag{8}$$

Applying a methodology similar to the one presented in the monograph [8], we construct the solution to the formulated initial-boundary-value problem (2)–(6) under the action of AMRP. For this purpose, we express the boundary condition (3) in the form of (1). As a result of these transformations, we obtain the expression for the component $H_z^{(n)}(r,t)$ of the magnetic field intensity vector under the influence of AMRP:

$$H_z^{(n)}(r,t) = \frac{k_0 H_0}{2} \sum_{i=0}^{2} \Big(B_{i1}^{(n)} e^{-(\beta_1 - i\omega)t} + B_{i2}^{(n)} e^{-(\beta_2 - i\omega)t} + B_{i3}^{(n)} e^{-(\beta_1 + i\omega)t} + B_{i4}^{(n)} e^{-(\beta_2 + i\omega)t} + B_{i5}^{(n)} e^{p_1 t} + B_{i6}^{(n)} e^{p_2 t} \Big) r^i. \tag{9}$$

Here, the expressions $B_{i1}^{(n)} \div B_{i6}^{(n)}$ depend on the geometric parameters of the SBMCS and the electrophysical parameters of the materials of its component cylinders, as well as on the time parameters of the AMRP. The values p_1, p_2 are the roots of the characteristic equation corresponding to (2). According to formula (6), the specific Joule heat density $Q^{(n)}(r,t)$ in n-th constituent cylinder of the SBMCS is written by the expression

$$Q^{(n)}(r,t) = \frac{1}{\sigma_n} \frac{k_0^2 H_0^2}{4} \sum_{i=1}^{2} \sum_{j=1}^{2} ij \sum_{l=1}^{20} C_{ijl}^{(n)} e^{\alpha_l t} r^{i+j-2}. \tag{10}$$

In this context, the expressions $C_{ijl}^{(n)}$, ($i = 1,2$, $j = 1,2$, $l = \overline{1,20}$) are described by appropriate combinations of expressions $B_{is}^{(n)}$ ($i = \overline{0,2}$, $s = \overline{1,6}$), α_l ($l = \overline{1,20}$) depend on the geometric parameters of the SBMCS, the frequency ω of electromagnetic oscillations, the parameters β_1 and β_2 of AMRP, and electrophysical characteristics σ_n, μ_n of the materials comprising of the SBMCS cylinder components.

4. Computer analysis of thermal modes

The calculations were performed for SBMCSs with radii of cylindrical surfaces $r_1 = 0.005$ m and $r_2 = 0.01$ m. Thermal regimes were investigated for the SBMCR made of non-ferromagnetic materials. Stainless alloy steel was chosen for the material of the internal solid component cylinder of the SBMCR, while the external hollow component cylinder of the SBMCR was made of copper. The electric and thermophysical characteristics of the materials of the constituent layers of the cylinder under consideration were selected from the reference book [9].

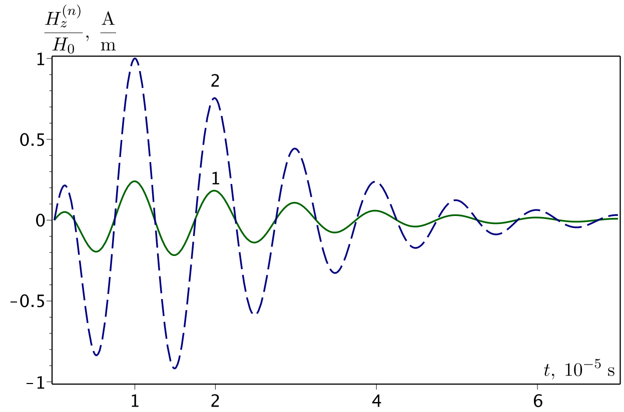

Fig. 1. Temporal variation of the axial component $H_z^{(n)}(r,t)$ of the magnetic field intensity vector on the surfaces $r = r_1$ (line 1) and $r = r_2$ (line 2).

The duration of AMRP was selected as $t_i = 100$ μs. This corresponds to the parameters of AMRP used for technological processing. The frequency of the carrier electromagnetic oscillations was taken as $\omega = 6.28 \cdot 10^5$ rad/s. This corresponds to the radio frequency range. With such a duration $t_i = 100$ μs and the specified angular frequency $\omega = 6.28 \cdot 10^5$ rad/s, ten periods of electromagnetic oscillations occur.

The results of the numerical analysis of the axial component $H_z^{(n)}(r,t)$ of the magnetic field intensity vector under the action of AMRP are shown in Figures 1, 2.

Figure 1 illustrates the variation over time of the axial component $H_z^{(n)}$ of the magnetic field intensity vector on the surface of the junction $r_1 = 0.005$ m (line 1) connecting the constituent cylinders of the SBMCR and on the outer surface $r_2 = 0.01$ m (line 2) of the SBMCR. Line 1 corresponds to the condition of ideal electromagnetic contact between the constituent cylinders of the SBMCR.

It has been established that the maximum values of the component $H_z^{(n)}(r,t)$ of the magnetic field intensity vector on the surface of the junction of the constituent cylinders are approximately 4 times smaller than those values on the outer surface of the SBMCR.

Figure 2 illustrates the distribution of the axial component $H_z^{(n)}$ of the magnetic field intensity vector along the radial variable of the constituent cylinders of the SBMCR at the moments of time $t = 0.05\ t_i$ (line 1), $t = 0.1\ t_i$ (line 2), and $t = 0.25\ t_i$ (line 3).

These moments in time correspond to the middle of the rising front, the maximum, and the middle of the falling front of the amplitude-modulated radio pulse signal.

It was found that the distribution of the magnetic field intensity vector component $H_z^{(1)}$ along the radial variable in the inner solid steel cylinder is close to linear. Accordingly, in the outer copper hollow cylinder, the distribution of the magnetic field intensity vector component $H_z^{(2)}$ is significantly nonlinear. It is established that the maximum values of the magnetic field intensity vector component $H_z^{(n)}$ are obtained at the time $t = 0.1\ t_i$.

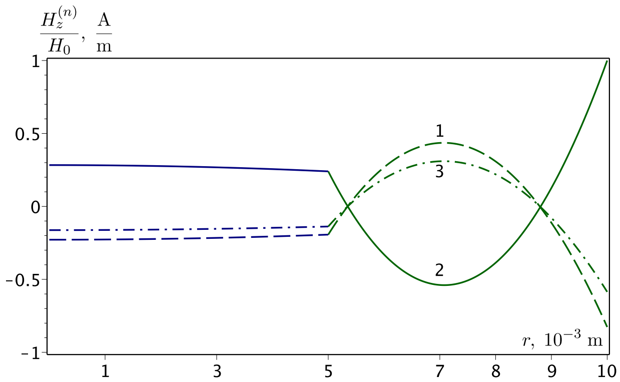

Fig. 2. Distribution of the axial component $H_z^{(n)}$ of the magnetic field intensity vector along the radial coordinate of the solid bimetallic cylinder at moments in time $t = 0.05\ t_i$ (line 1), $t = 0.1\ t_i$ (line 2), and $t = 0.25\ t_i$ (line 3).

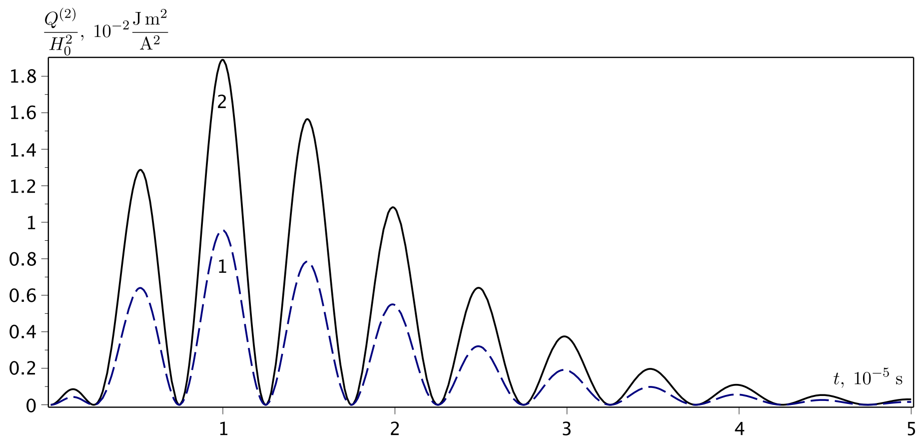

Fig. 3. Temporal variation of Joule heat $Q^{(2)}(r,t)$ in the external copper component cylinder on the surfaces $r = r_1$ (line 1) and $r = r_2$ (line 2).

In Figure 3, the variation over time of Joule heat $Q^{(2)}$ on the surface of the junction $r = r_1$ (line 1) connecting the constituent cylinders of the SBMCR and on the outer surface $r = r_2$ (line 2) of the SBMCR is depicted.

It has been determined that the maximum value of Joule heat $Q^{(2)}$ in the copper component cylinder on the outer surface $r = r_2$ is approximately 2 times greater than the corresponding value on the surface $r = r_1$ connecting the constituent cylinders of the SBMCR.

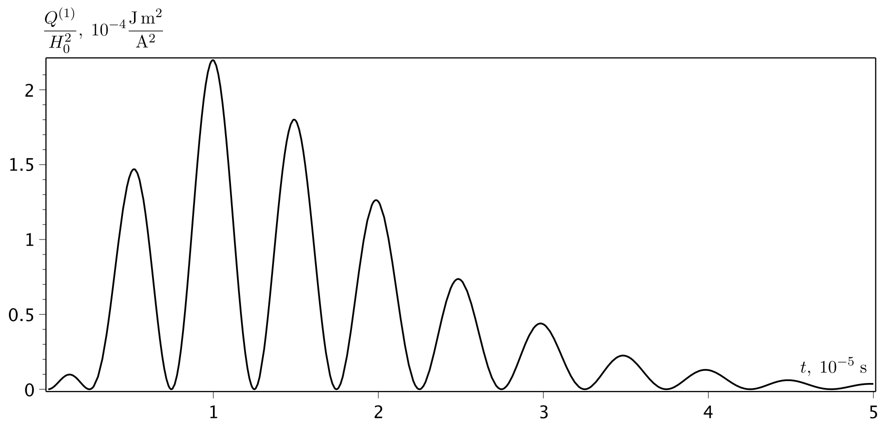

Fig. 4. Temporal variation of Joule heat $Q^{(1)}(r,t)$ in the inner steel composite cylinder on the surface $r = r_1$.

Figure 4 depicts the temporal variation of Joule heat $Q^{(1)}(r,t)$ in the internal steel component cylinder on the surface $r = r_1$ connecting the constituent cylinders of the SBMCR.

It is obtained that the maximum value of Joule heat $Q^{(1)}$ on the surface $r = r_1$ is about 100 times less than the following value of Joule heat $Q^{(2)}$ on the surface $r = r_2$.

The component $H_z^{(n)}$ of the magnetic field intensity vector and the Joule heat $Q^{(n)}$ acquire their maximum values at time $t = 0.1\,t_i$. This corresponds to the maximum of the function

$$\varphi(t) = \exp(-\beta_1 t) - \exp(-\beta_2 t),$$

which modulates the time dependence of AMRP.

Figure 5 shows the distribution of Joule heat $Q^{(1)}$ along the radial coordinate of the inner steel cylinder at times $t = 0.05\,t_i$ (line 1), $t = 0.1\,t_i$ (line 2), and $t = 0.25\,t_i$ (line 3).

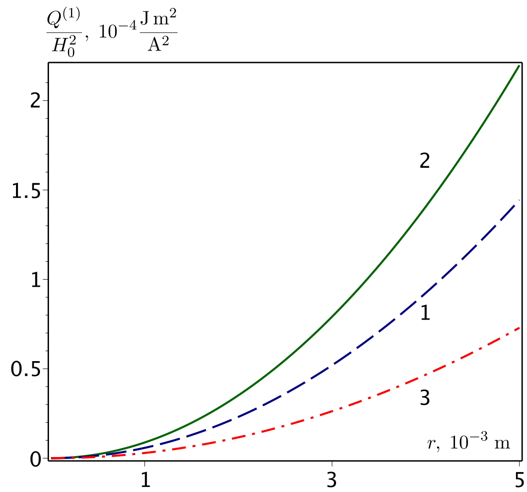

Fig. 5. Distribution of Joule heat $Q^{(1)}$ along the radial coordinate of the inner steel cylinder at times $t = 0.05\,t_i$ (line 1), $t = 0.1\,t_i$ (line 2) and $t = 0.25\,t_i$ (line 3).

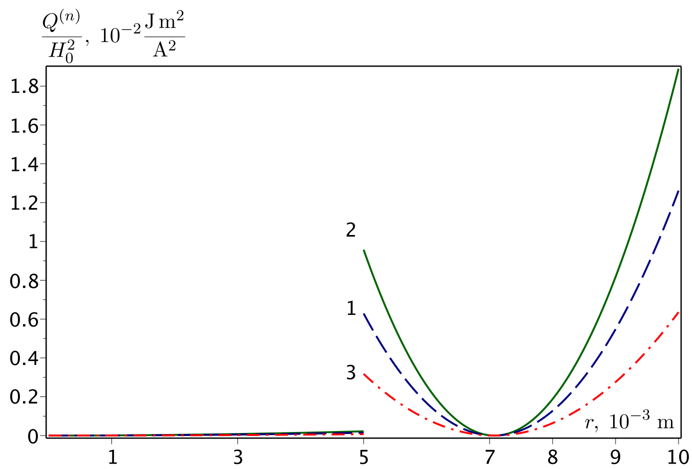

Fig. 6. Distribution of Joule heat along the radial coordinate of a solid bimetallic cylinder at times $t = 0.05\,t_i$ (line 1), $t = 0.1\,t_i$ (line 2) and $t = 0.25\,t_i$ (line 3).

It is obtained that the distribution of Joule heat $Q^{(1)}$ along the radial coordinate in a steel cylinder for the considered moments of time is close to linear. The maximum values are achieved at the time $t = 0.1\,t_i$ on the surface $r = r_1$.

Figure 6 shows the distribution of Joule heat $Q^{(n)}$ along the radial coordinate in the constituent cylinders of the SBMSC at times $t = 0.05\,t_i$ (line 1), $t = 0.1\,t_i$ (line 2), and $t = 0.25\,t_i$ (line 3).

It was found that on the surface of the joint of the component cylinders, the Joule heat undergoes jumps of the corresponding magnitude at the considered moments of time. It was found that the distribution of Joule heat in the steel component cylinder of the SBMCS is close to linear, while in the copper component cylinder of the SBMCS it is significantly nonlinear.

5. Conclusion

A physical-mathematical model has been introduced to explore the thermal behaviors within the constituent cylinders of the SBMCR under the influence of AMRP. The approach involved employing a quadratic approximation for the distributions of the axial component of the magnetic field intensity vector in the constituent cylinders of the SBMCR. This enabled the derivation of expressions for the aforementioned magnetic field component and the specific Joule heat density in the constituent cylinders of the SBMCR, presented in a form conducive to analytical convenience for computer analysis.

The time-dependent variations and radial coordinate distributions of the axial component of the magnetic field intensity vector and the specific Joule heat density in the constituent cylinders of the SBMCR were subjected to numerical analysis. Novel patterns, indicative of the thermal regimes within the constituent cylinders of the SBMCR during technological processing with amplitude-modulated radio pulses, were identified.

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Дослідження теплових процесів у суцільному біметалевому циліндрі за дії неусталеного електромагнітного поля

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У цій роботі представлено комплексну фізико-математичну модель, призначену для дослідження теплових процесів у суцільному біметалевому циліндричному стрижні, що зазнає впливу амплітудно-модульованого радіоімпульсу. Наведено вихідні співвідношення початково-крайової задачі електродинаміки для стрижня. Осьова компонента вектора напруженості магнітного поля вибрана як визначальна функція. Задача розв'язується шляхом апроксимації розподілів цієї визначальної функції в складових циліндрах за допомогою квадратичних поліномів по радіальній координаті. Описано вираз для осьової компоненти вектора напруженості магнітного поля та тепла Джоуля в складових циліндрах суцільного біметалевого стрижня під впливом амплітудно-модульованого радіоімпульсу. За допомогою чисельного аналізу досліджено зміну в часі та розподіл за радіальною координатою цих факторів у складових циліндрах біметалевого стрижня.

Ключові слова: *суцільний біметалевий циліндр; амплітудно-модульований радіоімпульс; осьова складова вектора напруженості магнітного поля; тепло Джоуля; теплові режими.*