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DESIGN OF DYNAMICAL SYSTEMS WITH POINT ATTRACTORS USING THE JACOBSTHAL-COLLATZ RECURRENT METHOD

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Abstract The work is devoted to the study of dynamical systems with point attractors by the recurrent method of transforming discrete data from the set of natural numbers, in the direction of increasing powers of two (direct Jacobsthal problem) and in the opposite direction (reverse Collatz problem). The idea of splitting the set *N* into separate non-overlapping subsets by Jacobsthal transformation of numbers was also expressed for the first time. It was established that this effect correlates with the regularities of Collatz-type sequences in the reverse direction of the transformation of the set *N* of initial numbers. It is shown that the number of segregation groups of the set N correlates with the number of periodic cycles of completion of Collatz sequences, plus the group of numbers that forms infinitely increasing Collatz sequences.

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Introduction and Problem Statement

For over 70 years now, the Collatz problem has been haunting mathematicians. It may seem that after an intense assault, some disappointment has set in. And how can we not recall Godel's incompleteness theorem, which is often cited in discussions about the existence of God. The Austrian mathematician Kurt Godel proved back in 1931 that in a sufficiently powerful formal system there are true statements that cannot be proven within the framework of the system itself.

If the Collatz problem, like the Jacobsthal numbers themselves, which are formed by four real numbers $\{G_0 = 2,\ G_1 = 1,\ t = 1,\ g = 2\}$ and obey the recurrence relation

$$G_{s+2} = t \cdot G_{s+1} + g \cdot G_s, \quad s \geq 0 \tag{1}$$

studied quite well (see also [4–8]), then from the point of view of the Collatz problem, they remained out of attention for a long time, until attention was drawn to them [9], and in [10], the fallacy of the conclusion [11] about the violation of the Collatz hypothesis was revealed with their help.

In essence, the Collatz problem is that no matter what number $q \in \mathrm{N}$ is greater than one, its transformation according to the algorithm

$$C^{+}_{3,q} = \ if \quad q \equiv 0 \mod 2 \ then \ \frac{q}{2} \ else \ C^{+}_{3,q} = 3q + 1 \tag{2}$$

equally ends in unity, or rather in a trivial periodic cycle $\ldots \to 1 \to 4 \to 2 \to 1 \to \ldots$. This cycle develops along a unit sequence on a binary basis $\{1 \cdot 2^n\}_{n=0}^{n=+\infty}$. Here, unit, as the smallest odd number, is the point to which the trajectories of the sequences $C^{+}_{3,q}$ converge or the so-called point attractor (*PA*=1).

In the case of the classical Collatz problem, the sequences $C^{+}_{3,q}$ are attracted to the unit attractor for the entire set of numbers $q \in \mathrm{N}$, as can be verified by calculations [7]. Here, the unit corresponds to the point of the sequence $\left\{1 \cdot 2^{n}\right\}_{n=0}^{n=+\infty}$, which is why it is also known as the root.

Review of Modern Information Sources on the Subject of the Paper

Despite the powerful many years of research on the problem, its results have only recently been applied in practice [12, 13]. These works showed the attractiveness and prospects of Collatz recurrent sequences not only for creating algorithms based on them for generating pseudorandom numbers, which is very important in design problems [1–4], but also showed, using the example of hash functions, their application in encryption problems.

In general, sequences $\left\{\theta \cdot 2^{n}\right\}_{n=0}^{n=+\infty}$ play an important role. With their help, even numbers $q \in \mathrm{N}$ can be doubled in the direction of increasing powers *n* of two 2^n; and halved in the reverse direction $n \to 0$. On the other hand, by establishing the rule of converting an even number $q_{even}(q_E)$ into an odd number $q_{odd}(q_O)$, it is possible to compose sequences $\left\{\theta \cdot 2^{n}\right\}_{n=0}^{n=+\infty}$ among themselves, $\mathrm{N} = \mathrm{N}_{odd}(\mathrm{N}_O) \cup \mathrm{N}_{even}(\mathrm{N}_E)$. $\theta \in \mathrm{N}_O$, $n \in \mathrm{N}_{\cup\{O\}}$, $q_O \in \mathrm{N}_O$. Before proving this, let's note the following.

In problem (2), odd numbers are converted to even numbers by the Collatz function $C^{+}_{3,q}$. However, a problem using a function with a mirror-symmetric sign $C^{-}_{3,q} = 3q - 1$ is known [14–16], but the algorithm is formulated similarly to (2):

$$C^{-}_{3,q} = \textit{ if } \quad q \equiv 0 \mod 2 \textit{ then } \frac{q}{2} \textit{ else } C^{-}_{3,q} = 3q - 1 \tag{3}$$

However, unlike (2), the sequences $C^{-}_{3,q}$ not only reach a periodic cycle ($\ldots \to 1 \to 2 \to 1 \to \ldots$) with a single point attractor *PA* = 1, but also periodic cycles with attractors *PA*=5,17. Moreover, if the initial numbers for which the process (3) *PA* = 1 is completed are: for a subset of ¥ 3,1, and *PA* = 5 (4a) for a subset of ¥ 3,5 and *PA* = 17 (4b) for a subset of ¥ 3,17:

$$\ldots \to 5 \to 7 \to \ldots \to 5 \to \ldots \text{ (a)} \qquad \ldots \to 17 \to 25 \to \ldots \to 17 \to \ldots \text{ (b)} \tag{4}$$

$$\mathrm{N} = \mathrm{N}_{3,1} \cup \mathrm{N}_{3,5} \cup \mathrm{N}_{3,17} \tag{5}$$

We will return to the analysis of (5). Here we only note that the effect of separation of the set ℕ (5) can be associated with the formation of trees isolated from each other with individual attractors *PA*. Similar patterns are manifested in the transformations of the numbers $C^{+}_{5,q} = 5q + 1$, $C^{-}_{7,q} = 7q - 1$ and $C^{+}_{181,q} = 181q + 1$ [15]. But unlike $C^{-}_{3,q} = 3q - 1$, when $\kappa \geq 5$ still constantly increasing sequences $C^{+(-)}_{\kappa \geq 5,q}$ are generated.

Thus, the following research objectives were formulated in this work:

– to investigate the direct Jacobsthal problem and its correlation with the reverse Collatz problem;

– to substantiate the effect of tree-like splitting of graphs of the set ℕ into separate, non-overlapping subsets.

–to investigate the correlation of the number of segregation groups of the set ℕ with the number of periodic and non-periodic cycles

Main Material Presentation

Consider the set ¥ of natural numbers

$$¥ \equiv \{ 1, 2, 3, 4, 5, 6, 7, \ldots, q, \ldots \} \quad , \tag{6}$$

and structure it in the form of a set of sequences $\left\{\theta \cdot 2^n\right\}_{n=0}^{n=+\infty}$

$$\begin{aligned} N = {} & 1\cdot2^{\circ}, 1\cdot2^{1}, 1\cdot2^{2}, 1\cdot2^{3}, 1\cdot2^{\circ}, ..., 1\cdot2^{n}, ..., 3\cdot2^{\circ}, 3\cdot2^{1}, 3\cdot2^{2}, 3\cdot2^{3}, 3\cdot2^{\circ}, ..., 3\cdot2^{n}, ..., 5\cdot2^{\circ}, 5\cdot2^{1}, \\ & 5\cdot2^{2}, 5\cdot2^{3}, 5\cdot2^{4}\,{}^{\circ}, ..., 5\cdot2^{n}, ..., 7\cdot2^{\wedge}\infty, 7\cdot2^{\circ}, 7\cdot2^{1}, 7\cdot2^{2}, 7\cdot2^{3}, 7\cdot2^{\circ}, ..., 7\cdot2^{n}, ... = \\ & = \left\{1\cdot 2^n\right\}_{n=0}^{n=+\infty}, \left\{3\cdot 2^n\right\}_{n=0}^{n=+\infty}, \left\{5\cdot 2^n\right\}_{n=0}^{n=+\infty}, \left\{7\cdot 2^n\right\}_{n=0}^{n=+\infty}, ..., \left\{\theta\cdot 2^n\right\}_{n=0}^{n=+\infty}, ..., \end{aligned} \tag{7}$$

where n ∈ ¥ ∪{0}, θ ∈ ¥ O, ¥ = ¥ E∪ ¥ O, ¥ E (¥ O) is the set of even (odd) numbers [16–22].

The set of sequences $\left\{\theta \cdot 2^n\right\}_{n=0}^{n=+\infty}$ has a binary basis, therefore, with their help, it is possible to implement the rule of doubling their members by doubling $2q$ in the direction $n \to +\infty$, and halving $q/2$ in the reverse direction $n \to 0$ of the change of exponent n.

On the other hand, at the points of each of the given sequences $\left\{\theta \cdot 2^n\right\}_{n=0}^{n=+\infty}$, in which the equality holds

$$\frac{\left\{\theta \cdot 2^n\right\}_{n=0}^{n=+\infty} \mu 1}{\kappa} = Integer\ , \tag{8}$$

odd Jacobsthal numbers are generated [16–22]

$$m(p)_{\kappa,\theta,n} = \frac{\theta \cdot 2^n \pm 1}{\kappa}. \tag{9}$$

Rule (9) with a minus sign generates the numbers $m_{\kappa,\theta,n}$, and rule (9) with a plus sign generates the numbers $p_{\kappa,\theta,n}$.

In Fig. 1, *a* to the right of the sequences $\left\{\theta \cdot 2^n\right\}_{n=0}^{n=+\infty}$, the integers $m_{3,\theta,n}$ with the parameter $\kappa = 3$ are written. These points, as for sequences $\left\{\theta \cdot 2^n\right\}_{n=0}^{n=+\infty}$ with other values of the parameter κ, i. e. with numbers $m(p)_{\kappa,\theta,n}$, correspond to the so-called nodes. In the direction of increasing powers *n* of two, other sequences are branched from the nodes, if the parameter θ of the so-called "parent" sequence $\left\{\theta \cdot 2^n\right\}_{n=0}^{n=+\infty}$ is not a multiple κ. Indeed, if $\kappa = 33$ and $\theta = 3$, then $m(p)_{33,3,n} = \frac{2\cdot 3^n \mu 1}{33} = \frac{1}{11}\cdot 2^n \mu \frac{1}{33}$ are fractional for all *n*.

Therefore, if κ is a multiple θ ($\theta = Integer\ \kappa = \theta$), then the numbers $m(p)_{\kappa,\theta,n}$ are fractional and branching nodes are not generated on the sequences $\left\{\theta \cdot 2^n\right\}_{n=0}^{n=+\infty}$. If necessary, multiples are highlighted in the figures with square brackets.

An illustration of the branching of sequences $\left\{\delta \cdot 2^n\right\}_{n=0}^{n=+\infty}$ from nodes is shown in Fig. 1, *b*.

$$\frac{\left\{1\cdot 2^n\right\}_{n=0}^{n=+\infty} - 1}{3} = Integer = m_{3,1,n} \quad \text{(a)} \quad \text{and} \quad \frac{\left\{1\cdot 2^n\right\}_{n=0}^{n=+\infty} + 1}{3} = Integer = p_{3,1,n} \quad \text{(b)}. \tag{10}$$

Here, according to rule (10a) at the points of the sequence $\left\{1\cdot 2^n\right\}_{n=0}^{n=+\infty}$, nodes with the numbers $m_{3,1,n} = 0, 1, 5, 21, 85, 341, ...$, are formed, from which other sequences $\left\{\delta \cdot 2^n\right\}_{n=0}^{n=+\infty}$ are generated. According to the rule opposite in sign (10b), nodes will be formed at the points of sequences $\left\{1\cdot 2^n\right\}_{n=0}^{n=+\infty}$ with values $p_{3,1,n} = 1, 3, 11, 43, 171, ...$. Thus, Fig. 1, *b* shows the so-called Jacobstal tree with the root sequence $\left\{1\cdot 2^n\right\}_{n=0}^{n=+\infty}$, which illustrates the branching model (10a).

n	$1\cdot2^n$	$m_{3,1,n}$	$3\cdot2^n$	$5\cdot2^n$	$m_{3,5,n}$	$7\cdot2^n$	$m_{3,7,n}$	$9\cdot2^n$	$11\cdot2^n$	$m_{3,11,n}$	$13\cdot2^n$	$m_{3,13,n}$	$15\cdot2^n$
0	$1\cdot2^0$	0	$3\cdot2^0$	$5\cdot2^0$		$7\cdot2^0$	2	$9\cdot2^0$	$11\cdot2^0$		$13\cdot2^0$	4	$15\cdot2^0$
1	$1\cdot2^1$		$3\cdot2^1$	$5\cdot2^1$	3	$7\cdot2^1$		$9\cdot2^1$	$11\cdot2^1$	7	$13\cdot2^1$		$15\cdot2^1$
2	$1\cdot2^2$	1	$3\cdot2^2$	$5\cdot2^2$		$7\cdot2^2$	9	$9\cdot2^2$	$11\cdot2^2$		$13\cdot2^2$	17	$15\cdot2^2$
3	$1\cdot2^3$		$3\cdot2^3$	$5\cdot2^3$	13	$7\cdot2^3$		$9\cdot2^3$	$11\cdot2^3$	29	$13\cdot2^3$		$15\cdot2^3$
4	$1\cdot2^4$	5	$3\cdot2^4$	$5\cdot2^4$		$7\cdot2^4$	37	$9\cdot2^4$	$11\cdot2^4$		$13\cdot2^4$	69	$15\cdot2^4$
5	$1\cdot2^5$		$3\cdot2^5$	$5\cdot2^5$	53	$7\cdot2^5$		$9\cdot2^5$	$11\cdot2^5$	117	$13\cdot2^5$		$15\cdot2^5$
6	$1\cdot2^6$	21	$3\cdot2^6$	$5\cdot2^6$		$7\cdot2^6$	149	$9\cdot2^6$	$11\cdot2^6$		$13\cdot2^6$	277	$15\cdot2^6$
7	$1\cdot2^7$		$3\cdot2^7$	$5\cdot2^7$	213	$7\cdot2^7$		$9\cdot2^7$	$11\cdot2^7$	469	$13\cdot2^7$		$15\cdot2^7$
8	$1\cdot2^8$	85	$3\cdot2^8$	$5\cdot2^8$		$7\cdot2^8$	597	$9\cdot2^8$	$11\cdot2^8$		$13\cdot2^8$	1109	$15\cdot2^8$
9	$1\cdot2^9$		$3\cdot2^9$	$5\cdot2^9$	853	$7\cdot2^9$		$9\cdot2^9$	$11\cdot2^9$	1877	$13\cdot2^9$		$15\cdot2^9$
10	$1\cdot2^{10}$	341	$3\cdot2^{10}$	$5\cdot2^{10}$		$7\cdot2^{10}$	2389	$9\cdot2^{10}$	$11\cdot2^{10}$		$13\cdot2^{10}$	4437	$15\cdot2^{10}$
11	$1\cdot2^{11}$		$3\cdot2^{11}$	$5\cdot2^{11}$	3413	$7\cdot2^{11}$		$9\cdot2^{11}$	$11\cdot2^{11}$	7509	$13\cdot2^{11}$		$15\cdot2^{11}$
12	$1\cdot2^{12}$	1365	$3\cdot2^{12}$	$5\cdot2^{12}$		$7\cdot2^{12}$	9557	$9\cdot2^{12}$	$11\cdot2^{12}$		$13\cdot2^{12}$	17749	$15\cdot2^{12}$

a

n	$\mathbf{1\cdot2^0}$	0			$141\cdot2^1$	**141**	$53\cdot2^3$		
0	$1\cdot2^1$		$3\cdot2^0$				$53\cdot2^2$		$213\cdot2^2$
1	$1\cdot2^2$	1	$3\cdot2^0$		$35\cdot2^0$	35	$53\cdot2^1$		$213\cdot2^1$
2	$1\cdot2^3$		3				$\mathbf{53\cdot2^0}$		**213**
3	$1\cdot2^4$	**5**	$5\cdot2^1$	$5\cdot2^2$	$5\cdot2^3$	$5\cdot2^4$	$5\cdot2^5$	$5\cdot2^6$	$5\cdot2^7$
4	$1\cdot2^5$				**13**				
5	$1\cdot2^6$				$13\cdot2^1$				
6	$1\cdot2^7$	21			$13\cdot2^2$	**17**	$17\cdot2^1$	$17\cdot2^2$	
7	$1\cdot2^8$	**85**	$85\cdot2^1$		$13\cdot2^3$		$\mathbf{11\cdot2^0}$		
8	$1\cdot2^9$				$13\cdot2^4$		$11\cdot2^1$	**7**	$7\cdot2^1$
9	$1\cdot2^{10}$	341	$341\cdot2^1$				$11\cdot2^2$		

b

Fig. 1. Illustration of the structuring of the ¥ as a of sequences (7) (a);

- illustration f of the formation of a Jacobsthal tree with nodes $m_{3,\theta,n}$ (b)

		$\mathbf{1\cdot2^0}$				$141\cdot2^1$	$141\cdot2^0$	$53\cdot2^3$	
		$1\cdot2^1$		$3\cdot2^0$				$53\cdot2^2$	
$1\cdot2^1$	$1\cdot2^0$	$1\cdot2^2$		$3\cdot2^0$		$35\cdot2^0$	$35\cdot2^0$	$53\cdot2^1$	
		$1\cdot2^3$		$3\cdot2^0$				$53\cdot2^0$	
		$1\cdot2^4$	$5\cdot2^0$	$5\cdot2^1$	$5\cdot2^2$	$5\cdot2^3$	$5\cdot2^4$	$5\cdot2^5$	$5\cdot2^6$
		$1\cdot2^5$				$13\cdot2^0$			
$21\cdot2^1$	$21\cdot2^6$	$1\cdot2^6$				$13\cdot2^1$			
		$1\cdot2^7$				$13\cdot2^2$	$17\cdot2^0$	$17\cdot2^1$	$17\cdot2^2$
		$1\cdot2^8$	$85\cdot2^6$	$85\cdot2^1$		$13\cdot2^3$		$11\cdot2^0$	
		$1\cdot2^9$				$13\cdot2^4$		$11\cdot2^1$	q=7
		$1\cdot2^{10}$	$341\cdot2^0$	$341\cdot2^0$				$11\cdot2^2$	

Fig. 2. Illustration of the CS_7 for $C_3^+ = 3q_{odd} + 1$

In the reverse direction $n \to 0$, Collatz sequences ($CS_{\kappa,q}$) are formed, in which branch nodes play the role of sequence $\{\theta \cdot 2^n\}_{n=0}^{n=+\infty}$ merging nodes according to the rule

$$\kappa \cdot m(p)_{\kappa,\theta,n} = \theta \cdot 2^n \pm 1, \tag{11}$$

as shown in Fig. 2 for the initial number q = 7, which is transformed by the function

$$C_3^+ = 3q_{odd} + 1 \tag{12}$$

Thus, in the Jacobsthal recurrent number model (9), known sequences CS_q are formed by transformation (11) from fragments of sequences $\{\theta \cdot 2^n\}_{n=0}^{n=+\infty}$. According to the Collatz conjecture, for numbers in the set ℕ transformed by the function (12), the sequences $CS_{\kappa,q}$ converge to unity (or the so-called unit point attractor PA = 1). Therefore, the conjecture can be further formulated as follows:

For all initial natural numbers, the sequences calculated by the algorithm $C^{+}_{3q+1} = \ if \ \ q \equiv 0 \bmod 2 \ \ then \ \frac{q}{2} \ \ else \ \ 3q+1$ *converge to the root sequence* $\left\{1 \cdot 2^{n}\right\}_{n=0}^{n=+\infty}$ *and end with the attractor PA* = 1.

However, there are known [9, 10, 23–28] transformations $\kappa \geq 3$ whose sequences $CS_{\kappa,q}$ can converge to non-identity point attractors, as:

PA=5.17 for $C_3^{-} = 3q-1$

$$cycle^{3q-1}_{1\to 1} = \{1 \to 1\}$$
$$cycle^{3q-1}_{5\to 5} = \{5 \to 7 \to 5\}$$
$$cycle^{3q-1}_{17\to 17} = \{17 \to 25 \to 37 \to 55 \to 41 \leftarrow 61 \to 91 \to 17\}$$

PA=13.17 for $C_5^{+} = 5q+1$

$$\mathrm{cycle}^{5q+1}_{1\to 1} = \{1 \to 3 \to 1\}$$
$$cycle^{5q+1}_{5\to 5} = \{13 \to 33 \to 83 \to 13\}$$
$$cycle^{5q+1}_{17\to 17} = \{17 \to 43 \to 27 \to 17\}$$

PA=27.35 for $C_{181}^{+} = 181q+1$

$$cycle^{181q+1}_{1\to 1} = \{\to 1 \to +\infty\}$$
$$cycle^{5q-1}_{5\to 5} = \{27 \to 611\}$$
$$cycle^{5q-1}_{17\to 17} = \{35 \to 99\}$$

In addition, when $\kappa \geq 5$, infinitely increasing sequences $CS_{\kappa,q}$ are formed, as for an initial number *q* = 7 with a transformation function $C_5^{+} = 5q_{odd}+1$. It is also important to note that in all cases, the total number of point attractors (trivial periodic cycles) of completion $CS_{\kappa,q}$ is equal to three.

Therefore, the purpose of the work is to substantiate the statement:

In the direction $n \to \infty$, numbers *q* are transformed according to the Jacobsthal algorithm

$$JS^{\mu}_{\kappa,q} = \begin{cases} 2q, \ q \equiv 0, \\ \dfrac{q \, \mu 1}{\kappa}, \ q \equiv 1, \end{cases}, \quad n \in \mathrm{N} \cup \{0\}. \tag{13}$$

In the direction $n \to \infty$, the $\dfrac{\left\{PA \cdot 2^{n}\right\}_{n=0}^{n=+\infty} \ \mu 1}{\kappa} = Integer$ transformation structures the set ℕ into separate independent subsets of numbers with isolated graph trees with root sequences $\left\{PA \cdot 2^{n}\right\}_{n=0}^{n=+\infty}$. In the reverse $n \to 0$ direction – along the tree , by the algorithm

$$C_{\kappa \cdot q \pm 1} = \ if \ \ q \equiv 0 \ \ \mathrm{mod}\, 2 \ \ then \ \frac{q}{2} \ \ else \ \ \kappa \cdot q \pm 1. \tag{14}$$

Collatz sequences $CS_{\kappa,q}$ are formed that converge to one of the independent point attractors *PA* and (or) grow infinitely. In other words, we will show that the consequence of the division of the set ℕ in the direction $n \to \infty$, into disjoint subsets, is the grouping of sequences $CS_{\kappa,q}$ in the form of a tree with isolated attractors *PA* (or pseudo attractor PA_{ps}) of root sequences $\left\{PA \cdot 2^{n}\right\}_{n=0}^{n=+\infty}$.

Periodic cycles with attractors $PA \neq 1$ are formed by the transformation functions of odd numbers $C_3^{-} = 3q-1$ and $C_5^{+} = 5q+1$. Therefore, for the function $C_3^{-} = 3q-1$ by the rule

$$\frac{\left\{\theta \cdot 2^{n}\right\}_{n=0}^{n=+\infty} + 1}{3} = Integer \tag{15}$$

we will construct a graph of branching sequences $\left\{\theta \cdot 2^n\right\}_{n=0}^{n=+\infty}$ at nodes with numbers $p_{3,\theta,n} = \dfrac{\theta \cdot 2^n + 1}{3}$ (Fig. 3, *a*, *b*), and for the function according to the rule $C_5^+ = 5q+1$

$$\frac{\left\{\theta \cdot 2^n\right\}_{n=0}^{n=+\infty} - 1}{5} = Integer \tag{16}$$

we construct a graph of branching sequences $\left\{\theta \cdot 2^n\right\}_{n=0}^{n=+\infty}$ at nodes with numbers $m_{5,\theta,n} = \dfrac{\theta \cdot 2^n - 1}{5}$ (Fig. 3, *c*, *d*).

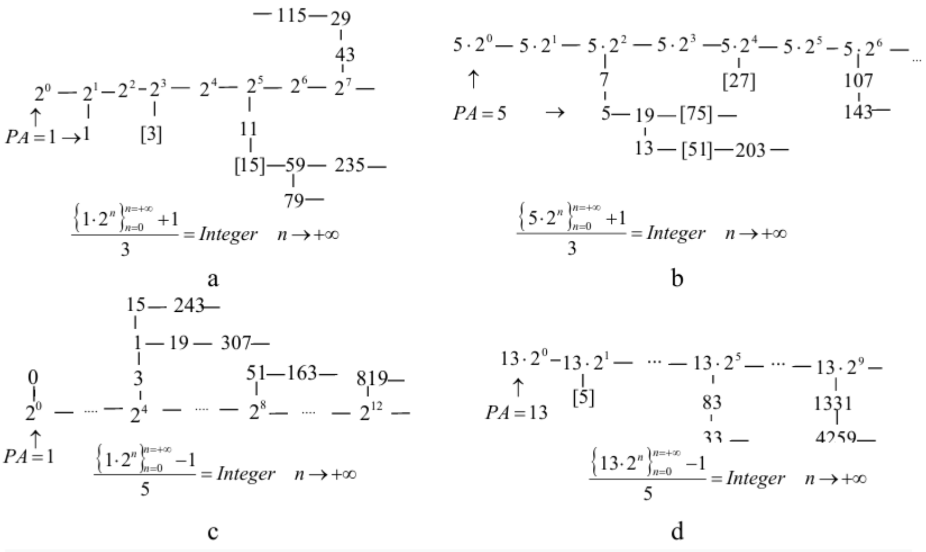


Fig. 3. *Illustration of fragments of the branching of the Jacobstal tree*

The fragment in Fig. 3, *a* shows that in the tree $\left\{1 \cdot 2^n\right\}_{n=0}^{n=+\infty}$ transformation (15) does not form nodes with odd numbers 5, 7, 9, 13, 17, 19, etc. Such numbers constitute two separate, isolated from ¥ sets, from which in the direction $n \to \infty$ two isolated from each other trees are formed, like the tree $\left\{5 \cdot 2^n\right\}_{n=0}^{n=+\infty}$ (Fig. 3, *b*) and the tree $\left\{17 \cdot 2^n\right\}_{n=0}^{n=+\infty}$. For the first 200 numbers this division of the set ¥ has the form [16, 17, 20]:

$cycle_1$	$cycle_5$	$cycle_{17}$	
3, 4, 6, 8, 11, 12, 15, 16, 22, 24, 28, 29, 30, 32, 39, 4, 44, 48, 53, 57, 58, 59, 60, 64, 65, 69, 71, 77, 78, 79, 85, 86, 87, 88, 95 ,96, 97, 101, 103, 105, 106, 113, 115, 116, 118, 120, 127, 128, 129, 130, 135, 137, 138, 141, 142, 145, 151, 154, 155, 156, 157, 158, 169, 170, 171, 172, 173, 174, 176, 183, 189, 190, 192, 193, 194	5, 7, 9, 10, 13, 14, 18, 19, 20, 26, 27, 35, 36, 38, 40, 47, 51, 52, 54, 56, 63, 70, 72, 75, 76, 80, 81, 89, 93, 94, 102, 104, 107, 108, 112, 119, 121, 124, 125, 126, 133, 139, 140, 143, 144, 149, 150, 152, 159, 159, 160, 161, 162, 167, 177, 178, 181, 186, 187, 188, 191, 199	17, 21, 23, 25, 31, 33, 34, 37, 41, 42, 45, 46, 49, 50, 55, 61, 62, 66, 67, 68, 73, 74, 82, 83, 84, 90, 91, 92, 98, 99, 100, 109, 110, 111, 114, 117, 122, 124, 123, 131, 132, 134, 136, 146, 147, 148, 153, 163, 164, 165, 166, 168, 175, 179, 180, 182, 184, 185, 195, 196, 197,200	(17)

Thus, by transformation (15), the set ℕ is divided into three non-overlapping sets, which form three trees: tree $\left\{1 \cdot 2^n\right\}_{n=0}^{n=+\infty}$, tree $\left\{5 \cdot 2^n\right\}_{n=0}^{n=+\infty}$, and tree $\left\{17 \cdot 2^n\right\}_{n=0}^{n=+\infty}$. For the model $C_3^+ = 3q+1$, there is no similar partition of the set ¥ , so all its numbers form nodes on only one tree $\left\{1 \cdot 2^n\right\}_{n=0}^{n=+\infty}$.

To confirm the idea of the stratification of the set ¥ , let us consider the regularities of tree $\left\{1 \cdot 2^n\right\}_{n=0}^{n=+\infty}$ branching according to rule (16), as shown in Fig. 3, *c*. A fragment of this tree shows that it does not form nodes with odd numbers 5, 7, 9, 11, 13, 17, 21, etc. As shown in Fig. 3, *c*, from this set of numbers, two trees isolated from each other are formed in the direction: tree $\{13 \cdot 2^n\}_{n=0}^{n=+\infty}$, tree $\{17 \cdot 2^n\}_{n=0}^{n=+\infty}$ and one tree $\{7 \cdot 2^n\}_{n=0}^{n=+\infty}$.

If on the tree $\{1 \cdot 2^n\}_{n=0}^{n=+\infty}$, tree $\{13 \cdot 2^n\}_{n=0}^{n=+\infty}$ and tree $\{17 \cdot 2^n\}_{n=0}^{n=+\infty}$ the sequences CS_q tend to periodic cycles, then on the tree $\{7 \cdot 2^n\}_{n=0}^{n=+\infty}$ the sequences CS_q grow to infinity, with the smallest odd number q_{odd} = 7, as shown in Fig. 4.

The tree shown in Fig. 4 is formed by transforming the initial number q = 116 in the direction $n \to +\infty$ according to the rule (16). Unlike the trees in Fig. 3, *a–d*, the tree in Fig. 4 is formed in the form of a parabola with a minimum at a point q = 7 and infinitely growing branches. The role of the root sequence in this case is played by $\{7 \cdot 2^n\}_{n=0}^{n=+\infty}$.

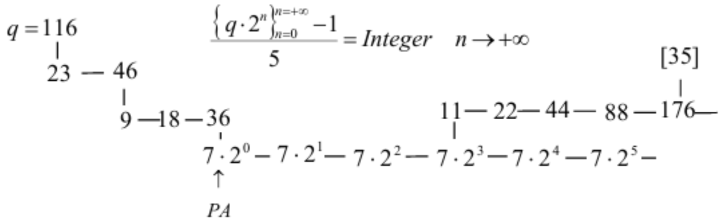


Fig. 4. Illustration of a Jacobsthal tree branching in a direction $n \to +\infty$ according to the rule (11) with a starting number q = 116

If in the direction $n \to +\infty$ the initial numbers are transformed according to the rule

$$\frac{\theta \cdot 2^n + 1}{5} = Integer \,, \tag{18}$$

then two isolated trees are formed, with root sequences $\{1 \cdot 2^n\}_{n=0}^{n=+\infty}$ and $\{9 \cdot 2^n\}_{n=0}^{n=+\infty}$ and with the minimum value of the odd number $q_{odd,\min}$ = 9. Therefore, the tree is formed similarly to the previous case.

Thus, in the reverse direction $n \to 0$, along the branches of the tree two types of sequences CS_q are formed. The first type of sequences CS_q are directed to periodic cycles with fixed attractors PA. Sequences CS_q of the second type without periodic cycles are formed in cases $\kappa \geq 5$ (Fig. 4), for which the role of the point attractor is played by the smallest odd number $q_{odd,min}$.

Let's consider the tree formed by the transformation

$$\frac{\theta \cdot 2^n + 1}{7} = Integer \,, \tag{19}$$

a fragment of which is shown in Fig. 6. Unlike the previous transformations, the tree in Fig. 6 is isolated from the root sequence $\{1 \cdot 2^n\}_{n=0}^{n=+\infty}$, since no nodes are formed on it. In this case, the sequence $CS_{\kappa,q}^{+(-)}$, which is formed by the reverse transformation of an arbitrary number $1 \cdot 2^n$, goes to unity, after which it grows infinitely. Therefore, in a tree of the type in Fig. 6, the sequence $\{1 \cdot 2^n\}_{n=0}^{n=+\infty}$ plays the role of an appendix.

Therefore, when $\kappa \geq 7$ sequences $CS_{7,q}$ are formed that are completely isolated from $\{1 \cdot 2^n\}_{n=0}^{n=+\infty}$ as in the case $7 \cdot q_0 - 1$ of , since for it the numbers $p_{1,1,n}$ are fractional and no nodes are formed.

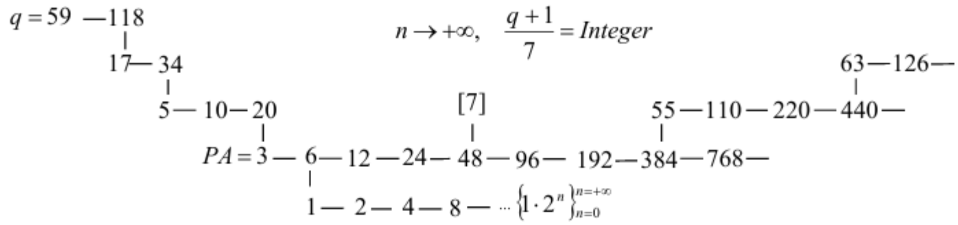


Fig. 5. Illustration of a Jacobsthal tree branching in a direction $n \longrightarrow +\infty$ according to the rule (14) with a starting number *q* = 59

However, in this case, the role of the root can be played by the sequence $\left\{3\cdot 2^n\right\}_{n=0}^{n=+\infty}$, for which the structure of nodes has the form:

$$\begin{array}{cccccccccc} p_{7,3,n}: & & 1 & & & 7 & & & 55 & \\ \downarrow & & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow \;,... \\ 3\cdot 2^n: & 3\cdot 2^0 & 3\cdot 2^1 & 3\cdot 2^2 & 3\cdot 2^3 & 3\cdot 2^4 & 3\cdot 2^5 & 3\cdot 2^6 & 3\cdot 2^7 & 3\cdot 2^8 \end{array} \tag{20}$$

Let's demonstrate this with an example of sequence formation $CS_{7,q=1006}$:

q=1006→**503**

↓

3520→1760→880→440→220→110→55→**29**

↓

384→192→96→48 →24→12→6→**3**

Isolated from the root $\left\{1\cdot 2^n\right\}_{n=0}^{n=+\infty}$ are the sequences $CS_{7,q}$ of the series of transformations $\kappa = 7+14\cdot n,\ \kappa = 7, 21, 35, 49, 63, 77, \ldots$, with fractional numbers $p_{7+14n,1,n}$ and $C_{(15+8n)q-1}$ $\kappa = 15+8i,\ \ i = 1, 2, 3, 4, \ldots$, with fractional numbers $p_{15+8i,1,n}$.

Now let us examine the frequency of occurrence of numbers in each of the isolated sets into which the set *N* is divided by the transformation (9). Analysis of the first million initial numbers showed that the sequences $CS_{3,q}^{+(-)}$, which are formed by the transformation of the function $C_3^- = 3q-1$, reach the attractor *PA* = 1 for 32.77 % of the initial numbers; reach the attractor for 32.34 % of the initial numbers, and the attractor for 34.90 % of the initial numbers. Thus, the set of initial numbers is divided into three equally probable by the attractor of the completion of the process . $C_3^- = 3q-1$

If in the transformation (9) both types of sequences $CS_{5,q}^{+(-)}$ are formed, as in the case of the function, then the attractor *PA* = 1 reaches ($2 \div 3$) % of numbers; the attractor *PA* = 13 reaches ($3 \div 4$) % of numbers; the attractor *PA* = 17 reaches ($1 \div 2$) % of numbers and for >90 % of numbers the sequences $CS_{5,q}^{+(-)}$ of which grow infinitely. The sequences $CS_{5,q}^{+(-)}$ formed by the function $C_5^- = 5q+1$ end in a periodic cycle with the attractor $PA = 1$ ($2 \div 3$) %), and for >90 % of numbers they also grow infinitely. The transformation $C_{181}^+ = 181q+1$ has a period $T_{181} = 56$ of node formation and two *PA* = 27.35 attractors and according to the laws of number transformation is similar to the model $C_5^- = 5q+1$. Statistical analysis has shown that out of the first 1000 numbers, the sequences CS_q reach the attractor *PA* = 1 for 10 numbers, the attractor *PA* = 27 for 7 numbers, the attractor *PA* = 35 for 9 numbers, and for 974 numbers the sequences CS_q grow infinitely.

Results and Discussion

Finally, we note the following. In the Collatz problem, the sequences $CS_{3,q}^{+}$ converge to unity. However, from the point of view of the sequence $\{1\cdot 2^{n}\}_{n=0}^{n=\infty}$, the point $n=0$ actually corresponds to the first element $1\cdot 2^{0}$ of the binary sequence. Therefore, the type conversion $3\cdot 1\cdot 2^{0}+1=4$ in the direction $n\to+\infty$ is controversial, while it is correct in the reverse direction $n\to 0$.

To eliminate the incorrectness, we construct a closed cycle from two trajectories of the Jacobsthal and Collatz transformations, where these two processes are separated by a dashed line (Fig. 5). Here the cycles are constructed for $\kappa=1\div 9$, $PA=1$, where the single arrows indicate the direction of the Jacobsthal transformation [18, 19, 21], and in the opposite direction the double Collatz transformations are carried out.

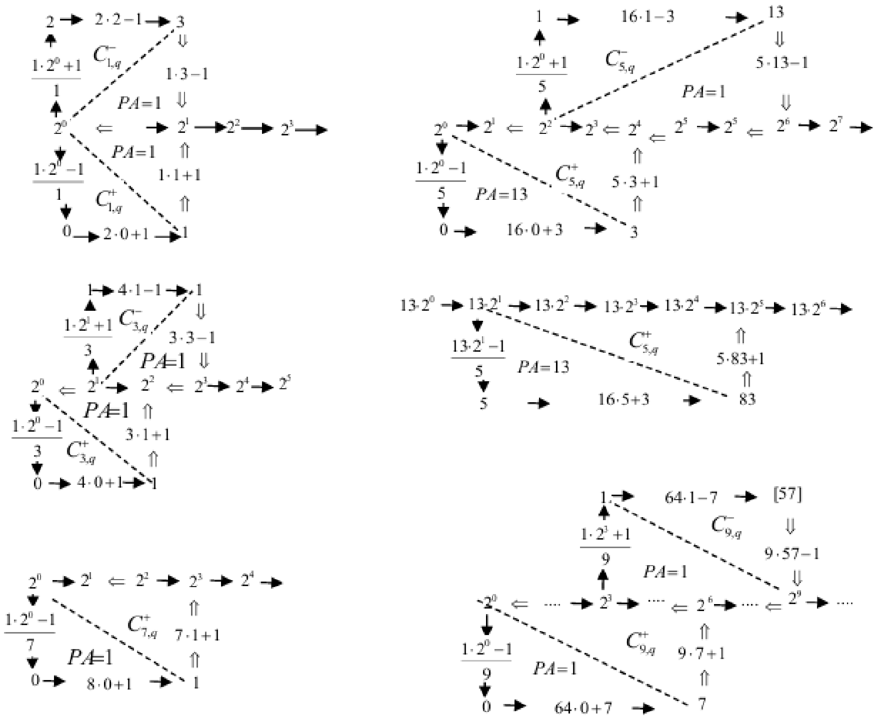


Fig. 6. Illustration of closed Jakobsthal – Collatz cycles

Therefore, if the sequence $CS_{\kappa,q}^{+(-)}$ has reached the attractor $PA\cdot 2^{0}$, then in the Jacobsthal – Collatz cycle model, a chain of the following transformations is carried out:

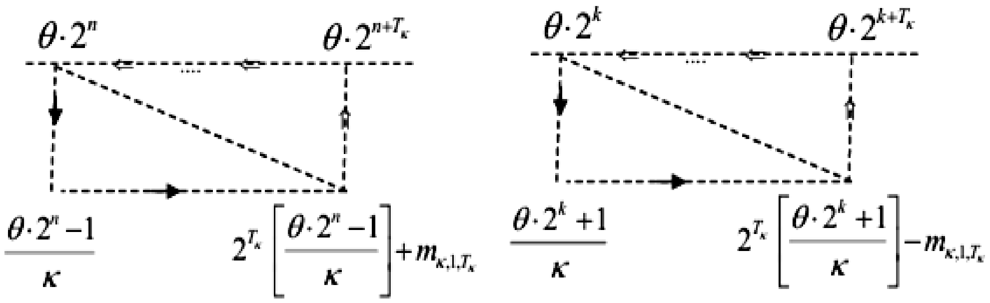
1. Condition is checked $\dfrac{PA\cdot 2^{0}-(+)1}{\kappa}=Integer$.

2. If the condition is fulfilled, it is assigned $\dfrac{PA\cdot 2^{0}-(+)1}{\kappa}=m_{\kappa,1,0}$ and a chain of cyclic transformations is performed:

$$m_{\kappa,1,0}\to 2^{T_\kappa}\cdot m_{\kappa,1,0}+(-)m_{\kappa,1,T_\kappa}\to\left[2^{T_\kappa}\cdot m_{\kappa,1,0}+\cdot m_{\kappa,1,T_\kappa}\right]\cdot\kappa+(-)1=\theta\cdot 2^{n+T_\kappa}\qquad(21)$$

3. If $\dfrac{PA\cdot 2^{0}-(+)1}{\kappa}\neq Integer$, then the number $PA\cdot 2^{0}$ is doubled v times until the equality $\dfrac{2^{\nu}\cdot PA\cdot 2^{0}-(+)1}{\kappa}\neq Integer=m(p)_{\kappa,\theta,n(T_\kappa)}$ is fulfilled, that is, the first odd number $m(p)_{\kappa,\theta,n(T_\kappa)}$ is generated, which in the future, according to clause 2, will form a cycle.

Fig. 7 shows generalized schemes of cycle formation for the most typical cases, which can be generalized as shown in Fig. 7, *a*. Here, on an arbitrarily taken sequence $\{\theta \cdot 2^n\}_{n=0}^{n=+\infty}$, a point $\theta \cdot 2^n$ is arbitrarily chosen. If the equality $\frac{\theta \cdot 2^n - 1}{\kappa} = Integer$ holds for it, then in the direction $n \to +\infty$, from this point a node with the number $m_{\kappa,\theta,n} = \frac{\theta \cdot 2^n - 1}{\kappa}$ is generated, which is subsequently transformed according to rule (21): $m_{\kappa,\theta,n+T_0} = \frac{2^{T_\kappa}}{\kappa}\left(\frac{\theta \cdot 2^0 - 1}{\kappa}\right) + m_{\kappa,1,T_0}$.



a b

Fig. 7. Generalized models of closed Jacobsthal – Collatz cycles with numbers $m_{\kappa,\theta,n}$ (a) and $p_{\kappa,\theta,k}$ (b) of branch nodes

This completes the Jacobsthal link. Next, the odd number $m_{\kappa,\theta,n+T_0}$ according to algorithm (16) is converted into an even number of the sequence $\{\theta \cdot 2^n\}_{n=0}^{n=+\infty}$, which after a period again acquires the previous value $\theta \cdot 2^n$. Thus, a periodic cycle is formed regardless of the parameter κ. To confirm the Jacobsthal-Collatz periodic cycle model in Fig. 7, we will substantiate the formula for the period from it [16]:

$$\left[\frac{2^{T_\kappa}}{\kappa}\left(\theta \cdot 2^n - 1\right) + m_{\kappa,1,T_\kappa}\right] \cdot \kappa + 1 = \theta \cdot 2^{n+T_\kappa} \Rightarrow 2^{T_\kappa} = \kappa \cdot m_{\kappa,1,T_\kappa} + 1 \quad (22)$$

If for the root sequence $\{1 \cdot 2^n\}_{n=0}^{n=\infty}$ at the first point $\frac{1 \cdot 2^0 - (+)1}{\kappa} \neq 0$, then the attractor is still *PA* = 1. After all, along $\{1 \cdot 2^n\}_{n=0}^{n=\infty}$ the process of dividing an even number in half is completed when *PA* = 1, after which the first point of fulfillment of the condition $\frac{1 \cdot 2^0 - (+)1}{\kappa} = Integer$ is reached by doubling.

Conclusions

According to the rules (9,10), in the direction $n \to \infty$ the set *N* of numbers is divided into sets isolated from each other, each of which forms a Jacobsthal tree with a root sequence $PA \cdot 2^n$ and a point attractor $PA \cdot 2^0$ of a trivial periodic cycle, to which Collatz sequences $CS_{\kappa,q}^{+(-)}$ go in the reverse $n \to 0$ direction. At $\kappa \geq 5$, the number of sets increases by one, for numbers of which the sequences do not end with a periodic cycle, but grow infinitely. For the transformation $\frac{\theta \cdot 2^n - 1}{3} = Integer$, there is no partition of the set *N*, therefore the sequences $CS_{\kappa,q}^{+(-)}$ converge to a single attractor $PA = 1 \cdot 2^n$ for the entire set *N* of numbers. The transformation of numbers N according to the rules $\frac{\theta \cdot 2^n \pm 1}{\kappa} = Integer$ in the direction $n \to +\infty$ is primary, which determines the regularities of the formation of the sequence in the reverse direction $CS_{\kappa,q}^{+(-)}$.

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ПРОЄКТУВАННЯ ДИНАМІЧНИХ СИСТЕМ ІЗ ТОЧКОВИМИ АТРАКТОРАМИ З ВИКОРИСТАННЯМ РЕКУРЕНТНОГО МЕТОДУ ЯКОБСТАЛЯ – КОЛЛАТЦА

Отримано: Серпень 14, 2025 / Переглянуто: Серпень 20, 2025 / Прийнято: Вересень 15, 2025

Анотація. Досліджено динамічні системи із точковими атракторами рекурентним методом перетворення дискрентних даних із множини натуральних чисел у напрямку зростання степеня двійки (пряма задача Якобсталя) і у зворотному напрямку (реверсна задача Коллатца). Вперше висловлено ідею розшарування перетворенням чисел Якобсталя множини ℕ на окремі підмножини, що не перекриваються. Встановлено, що цей ефект корелює із закономірностями послідовностей типу Коллатца в реверсному напрямку перетворення множини ℕ початкових чисел. Показано, що кількість груп сегрегації множини ℕ корелює із кількістю періодичних циклів завершення послідовностей Коллатца, плюс група чисел, що формує послідовності Коллатца, які безмежно зростають.

Ключові слова: задача Коллатца, рекурентні послідовності, числа Якобсталя, точковий атрактор, 2020 Mathematics Subject Classification: 37P99;11Y16; 11A51; 11-xx; 11Y50.

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