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The accuracy of the Cayley transform method for an evolution equation with a fractional derivative

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The initial value problem for a differential equation with a fractional derivative and a positive definite operator coefficient in a Hilbert space is considered. The exact solution involves the solving operator (expressed as an infinite series incorporating the Cayley transform of the operator coefficient, and certain polynomials of the independent variable, which is known as the Laguerre–Cayley polynomials) and the convolution integral of the solving operator with the right-hand side of the equation. The approximate solution is expressed through the partial sum of the first N terms of this series. Then, we obtain error estimates by taking into account certain smoothness properties of the input data: at first, we prove the power rate of convergence depending on the discretization parameter N in the case of a finitely differentiable right-hand side of the equation, and next, we prove the exponential rate of convergence if the right-hand side is analytic in some sense.

Keywords: *initial value problem (IVP); Hilbert space; Mittag–Leffler function; Cayley transform; algorithm without saturation of accuracy; exponentially convergent algorithm.*

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1. The problem statement

In modern science and its applications, the modeling of complex processes characterized by memory effects, fractal structures, nonlocality, nonlinear and anomalous dynamics is of current relevance. Traditional approaches to describing such phenomena often lack sufficient flexibility, whereas the framework of fractional integro-differentiation enables the construction of more adequate models of real-world processes. This makes it particularly relevant in such fields as materials science, biology, finance, signal processing, and others [1–4]. The development of numerical algorithms and modern computational technologies has significantly simplified the practical application of fractional calculus, further stimulating its widespread use in both theoretical and applied research.

This topic is addressed in [5–7], where approximate grid methods for solving boundary value problems for differential equations with fractional derivatives are developed, and their accuracy is studied considering the influence of boundary conditions. In contrast to the aforementioned publications, the goal of the present paper is to solve the abstract equation with an integro-differential fractional-order derivative using the Cayley transform method [8–10]. Theoretical analysis, numerical calculations, and their comparison with the results of [11] indicate that the Cayley transform method is an effective approach for solving such problems.

We consider the Cauchy problem for an evolution equation with a sectorial operator in a Banach space with a fractional derivative in the formulation of [11]:

$$\begin{gathered} \partial_t u + \partial_t^{-\alpha} A u = f(t), \quad t > 0, \\ u(0) = u_0, \end{gathered} \tag{1}$$

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where $\partial_t u = du/dt$ and

$$(\partial_t^{-\alpha} u)(t) = \begin{cases} \dfrac{d}{dt}\displaystyle\int_0^t \frac{(t-s)^\alpha}{\Gamma(1+\alpha)} u(s)\,ds & \text{if } \ -1 < \alpha \leqslant 0; \\ \displaystyle\int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} u(s)\,ds & \text{if } \ 0 < \alpha < 1. \end{cases}$$

The case $-1 < \alpha < 0$ corresponds to the modeling of a subdiffusion process when the mean square displacement of the diffusing particles is proportional to $t^{1+\alpha}$. For $\alpha = 0$, equation (1) transforms into the classical parabolic equation when the mean square displacement of the diffusing particles is proportional to t. The case $0 < \alpha < 1$ is of interest for the viscoelasticity problems.

With the help of the Mittag–Leffler function [12]

$$E_\mu(z) = \sum_{j=0}^{\infty} \frac{z^j}{\Gamma(1+j\mu)}, \quad z \in \mathbb{C} \quad (\mu > 0),$$

the solution of problem (1) can be formally presented as follows [1]:

$$u(t) = U(t)u_0 + \int_0^t U(t-s) f(s)\,ds \tag{2}$$

with

$$U(t) = E_{1+\alpha}(-t^{1+\alpha}A) = \sum_{j=0}^{\infty} \frac{1}{\Gamma\big(1+j(1+\alpha)\big)} (-t^{1+\alpha}A)^j. \tag{3}$$

The alternative form of the solving operator $U(t)$ in (3) is found. Applying the Cayley transform of the operator A:

$$Q = A(I+A)^{-1}, \quad A = (I-Q)^{-1}Q,$$

one can obtain

$$U(t) = E_{1+\alpha}\left(-t^{1+\alpha}(I-Q)^{-1}Q\right) = \sum_{j=0}^{\infty} \frac{1}{\Gamma\big(1+j(1+\alpha)\big)} \left(-t^{1+\alpha}(I-Q)^{-1}Q\right)^j. \tag{4}$$

Next, we formally replace the operator Q in (4) with a scalar q and provide the following definition.

Definition 1. *The functions $p_n^{(\alpha)}(t^{1+\alpha})$ produced after expanding the function $E_{1+\alpha}\big(-\frac{q}{1-q}t^{1+\alpha}\big)$ into a Maclaurin series in powers of q:*

$$E_{1+\alpha}\left(-\frac{q}{1-q}t^{1+\alpha}\right) = \sum_{j=0}^{\infty}\left(-\frac{q}{1-q}t^{1+\alpha}\right)^j \frac{1}{\Gamma\big(1+j(1+\alpha)\big)} = \sum_{n=0}^{\infty} p_n^{(\alpha)}(t^{1+\alpha})q^n,$$

$$p_n^{(\alpha)}(t^{1+\alpha}) = \frac{1}{n!}\frac{\partial^n}{\partial q^n} E_{1+\alpha}\Big(-\frac{q}{1-q}t^{1+\alpha}\Big)\Big|_{q=0}, \quad \alpha > -1,$$

are called the Laguerre–Cayley functions, and the polynomials $p_n^{(\alpha)}(x)$ are called the Laguerre–Cayley polynomials.

From Definition 1, it is easy to derive the explicit representation of the Laguerre–Cayley functions:

$$p_n^{(\alpha)}(t^{1+\alpha}) = \frac{1}{n!}\frac{\partial^n}{\partial q^n} E_{1+\alpha}\left(-\frac{q}{1-q}t^{1+\alpha}\right)\Big|_{q=0} = \frac{1}{n!}\sum_{j=0}^{\infty} \frac{(-1)^j t^{j(1+\alpha)}}{\Gamma\big(1+j(1+\alpha)\big)} \frac{d^n}{dq^n}\big(q^j(1-q)^{-j}\big)\Big|_{q=0}$$
$$= \sum_{r=0}^{n-1} \frac{(-1)^{r+1} C_{n-1}^r}{\Gamma\big(1+(r+1)(\alpha+1)\big)} t^{(r+1)(\alpha+1)}, \quad n = 1, 2, \ldots, \quad p_0^{(\alpha)}(t^{\alpha+1}) \equiv 1.$$

Then the Laguerre–Cayley polynomials $p_n^{(\alpha)}(x)$ can also be presented explicitly:

$$p_n^{(\alpha)}(x) = \sum_{r=0}^{n-1} \frac{(-1)^{r+1} C_{n-1}^r}{\Gamma\big(1+(r+1)(\alpha+1)\big)} x^{r+1}, \quad n = 1, 2, \ldots, \quad p_0^{(\alpha)}(x) \equiv 1, \tag{5}$$

where $C_n^k = \frac{n!}{k!(n-k)!}$ are the binomial coefficients.

With Definition 1 in mind, we can write solution (2) as follows:

$$u(t) = U(t)u_0 + \int_0^t U(s)f(t-s)\,ds \tag{6}$$

with the solving operator

$$U(t) = \sum_{n=0}^{\infty} p_n^{(\alpha)}(t^{1+\alpha})Q^n, \quad Q = A(I+A)^{-1}. \tag{7}$$

The Cayley transform method consists in taking a finite sum of the series in formulas (6) and (7) as an approximate solution of problem (1), namely:

$$u_N(t) = U_N(t)u_0 + \int_0^t U_N(s)f(t-s)\,ds, \tag{8}$$

where

$$\begin{gathered} U_N(t) = \sum_{n=0}^{N} p_n^{(\alpha)}(t^{1+\alpha})Q^n, \quad Q = A(I+A)^{-1}, \\ \int_0^t U_N(s)f(t-s)\,ds = \sum_{n=0}^{N} Q^n \int_0^t p_n^{(\alpha)}(s^{1+\alpha})f(t-s)\,ds. \end{gathered} \tag{9}$$

We consider the case of a Hilbert space H with an inner product (u,v) $\forall u,v \in H$ and associate norm $\|u\| = \sqrt{(u,u)}$. Let A be a self-adjoint positive definite operator, and let $\{\varphi_i\}_{i=1}^{\infty} \subset H$ be an orthonormal basis in H, formed by the eigenvectors of the operator A, corresponding to the eigenvalues λ_i, $i = 1, 2, \ldots$, such that $0 < \lambda_1 \leqslant \lambda_2 \leqslant \ldots$.

In the present paper, we investigate the accuracy of method (8), (9) with various assumptions regarding the initial vector u_0 and the right-hand side $f(t)$.

2. The Cayley transform method without accuracy saturation

First of all, we need the following auxiliary statement.

Lemma 1. *Let $\alpha \geqslant 2\beta$ and $\beta > 0$. Then the following inequality holds true:*

$$\max_{\lambda>0} \left(\frac{\lambda}{\lambda+1}\right)^{\alpha} \lambda^{-\beta} \leqslant \frac{e^{-\beta}(2\beta)^{\beta}}{\alpha^{\beta}}.$$

Proof. We consider the function $\varphi(\lambda) = \left(\frac{\lambda}{\lambda+1}\right)^{\alpha}\lambda^{-\beta}$, $\lambda > 0$. Since $\varphi'(\lambda) = \frac{\lambda^{-\alpha-\beta-1}}{(\lambda+1)^{2\alpha+1}}\big((\alpha-\beta)-\beta\lambda\big)$, then under the assumption $\alpha \geqslant 2\beta$ it follows that $\alpha - \beta \geqslant \alpha/2$, and therefore,

$$\max_{\lambda>0}\varphi(\lambda) = \varphi\left(\frac{\alpha-\beta}{\beta}\right) = \left(1-\frac{\beta}{\alpha}\right)^{\alpha}\left(\frac{\alpha-\beta}{\beta}\right)^{-\beta} = \exp\left(\alpha\ln\left(1-\frac{\beta}{\alpha}\right)\right)\frac{\beta^{\beta}}{(\alpha-\beta)^{\beta}} \leqslant \frac{e^{-\beta}(2\beta)^{\beta}}{\alpha^{\beta}}.$$

The lemma is proved. ∎

Remark 1. From here on, we assume that the Laguerre–Cayley functions $p_n^{(\alpha)}(t^{1+\alpha})$ satisfy the condition

$$\left|p_n^{(\alpha)}(t^{1+\alpha})\right| \leqslant C(t)n^{\gamma}, \quad t \in [0;T], \tag{10}$$

where $\gamma \in \mathbb{R}$ and $C(t) > 0$ is independent of n. The basis for this assumption is a significant number of calculations performed using the computer algebra system Maple. For example, inequality (10) follows from the convergence of the next series:

$$\sum_{n=1}^{\infty} p_n^{(-3/4)}(7/8) = -1.1428\ldots, \qquad \sum_{n=1}^{\infty} n p_n^{(-1/2)}(1) = -0.5641\ldots,$$

$$\sum_{n=1}^{\infty} n^2 p_n^{(-3/4)}(1) = -0.3123\ldots, \qquad \sum_{n=1}^{\infty} n^3 p_n^{(-8/9)}(3/2) = 0.2061\ldots$$

(see also the results of [13]).

First, we will examine the accuracy of the Cayley transform method in the case of a homogeneous equation. In this case, formulas (6) and (8) for the exact and approximate solutions of problem (1), respectively, take the form

$$u(t) = U(t)u_0 = \sum_{n=0}^{\infty} p_n^{(\alpha)}(t^{1+\alpha})Q^n u_0, \tag{11}$$

$$u_N(t) = U_N(t)u_0 = \sum_{n=0}^{N} p_n^{(\alpha)}(t^{1+\alpha})Q^n u_0. \tag{12}$$

The domain of the operator A^σ is denoted by $D(A^\sigma)$.

Theorem 1. *Let $f(t) \equiv 0$, the Laguerre–Cayley functions $p_n^{(\alpha)}(t^{1+\alpha})$ satisfy condition (10), and $u_0 \in D(A^\sigma)$, $\sigma > \gamma + 1$. Then the Cayley transform method (12) is a method without accuracy saturation, and the following error estimate holds true:*

$$\|u(t) - u_N(t)\| \leqslant C(t)\frac{(2\sigma)^\sigma e^{-\sigma}}{(\sigma-\gamma-1)}\frac{1}{N^{(\sigma-\gamma-1)}}\|A^\sigma u_0\|. \tag{13}$$

Proof. From formulas (11) and (12), we obtain

$$\begin{aligned}
\|u(t) - u_N(t)\|^2 &= \left\| \sum_{n=N+1}^{\infty} p_n^{(\alpha)}(t^{1+\alpha})Q^n A^{-\sigma}A^\sigma u_0 \right\|^2 \\
&= \left\| \sum_{n=N+1}^{\infty} p_n^{(\alpha)}(t^{1+\alpha}) \sum_{i=1}^{\infty} \left(\frac{\lambda_i}{1+\lambda_i}\right)^n \lambda_i^{-\sigma}(A^\sigma u_0, \varphi_i)\varphi_i \right\|^2 \\
&= \left\| \sum_{i=1}^{\infty} \left[\sum_{n=N+1}^{\infty} p_n^{(\alpha)}(t^{1+\alpha}) \left(\frac{\lambda_i}{1+\lambda_i}\right)^n \lambda_i^{-\sigma}(A^\sigma u_0, \varphi_i) \right] \varphi_i \right\|^2 \\
&= \sum_{i=1}^{\infty} \left| \sum_{n=N+1}^{\infty} p_n^{(\alpha)}(t^{1+\alpha}) \left(\frac{\lambda_i}{1+\lambda_i}\right)^n \lambda_i^{-\sigma}(A^\sigma u_0, \varphi_i) \right|^2 \\
&\leqslant \sum_{i=1}^{\infty} \left|(A^\sigma u_0, \varphi_i)\right|^2 \left[\sum_{n=N+1}^{\infty} \left|p_n^{(\alpha)}(t^{1+\alpha})\right| \left(\frac{\lambda_i}{1+\lambda_i}\right)^n \lambda_i^{-\sigma} \right]^2 .
\end{aligned}$$

Applying Lemma 1 with $\alpha = n$ and $\beta = \sigma$, we have

$$\begin{aligned}
\|u(t) - u_N(t)\|^2 &= \left[\sum_{n=N+1}^{\infty} C(t)n^\gamma \left(\frac{2\sigma}{e}\right)^\sigma \frac{1}{n^\sigma} \right]^2 \sum_{i=1}^{\infty} \left|(A^\sigma u_0,\, \varphi_i)\right|^2 \\
&\leqslant C^2(t)\left(\frac{2\sigma}{e}\right)^{2\sigma} \left[\sum_{n=N+1}^{\infty} \frac{1}{n^{\sigma-\gamma}} \right]^2 \|A^\sigma u_0\|^2 \leqslant C^2(t)\left(\frac{2\sigma}{e}\right)^{2\sigma} \left[\int_N^\infty \frac{dx}{x^{\sigma-\gamma}} \right]^2 \|A^\sigma u_0\|^2 \\
&= C^2(t)\frac{(2\sigma)^{2\sigma}e^{-2\sigma}}{(\sigma-\gamma-1)^2}\frac{1}{N^{2(\sigma-\gamma-1)}}\|A^\sigma u_0\|^2,
\end{aligned}$$

which yields estimate (13). The theorem is proved. ■

Next, we analyze the accuracy of the Cayley transform method for the inhomogeneous equation with zero initial condition. To this end, we write formulas (6) and (8) for the exact and approximate solutions of problem (1), respectively:

$$u(t) = \int_0^t U(s)f(t-s)\,ds = \int_0^t \sum_{n=0}^{\infty} p_n^{(\alpha)}(s^{1+\alpha})Q^n f(t-s)\,ds, \quad Q = A(I+A)^{-1}, \tag{14}$$

$$u_N(t) = \int_0^t U_N(s)f(t-s)\,ds = \int_0^t \sum_{n=0}^{N} p_n^{(\alpha)}(s^{1+\alpha})Q^n f(t-s)\,ds, \quad Q = A(I+A)^{-1}. \tag{15}$$

We now proceed to prove the statement.

Theorem 2. *Let $u_0 = 0$, the Laguerre–Cayley functions $p_n^{(\alpha)}(t^{1+\alpha})$ satisfy inequality (10) with $0 < C(t) \leqslant C$ $\forall t \in [0;T]$, and the vector function $f(t)$ meet the conditions*

$$f(t) \in D(A^\sigma) \quad \forall t \in [0;\, T],\ \sigma > \gamma + 1; \qquad \int_0^t \|A^\sigma f(s)\|^2 ds < \infty.$$

Then the Cayley transform method (15) is a method without accuracy saturation, and the error estimate holds true:

$$\|u(t) - u_N(t)\| \leqslant \sqrt{t}\, C \frac{(2\sigma)^\sigma e^{-\sigma}}{\sigma - \gamma - 1} \frac{1}{N^{\sigma-\gamma-1}} \left\{ \int_0^t \|A^\sigma f(s)\, ds\|^2 ds \right\}^{1/2}. \tag{16}$$

Proof. From formulas (14) and (15), we obtain the error representation and the chain of inequalities

$$\begin{aligned}
\|u(t) - u_N(t)\|^2 &= \left\| \int_0^t \big(U(s) - U_N(s)\big) f(t-s)\, ds \right\|^2 \\
&= \left\| \int_0^t \sum_{n=N+1}^{\infty} p_n^{(\alpha)}(s^{1+\alpha})\, Q^n A^{-\sigma} A^\sigma f(t-s)\, ds \right\|^2 \\
&= \left\| \int_0^t \sum_{n=N+1}^{\infty} p_n^{(\alpha)}(s^{1+\alpha}) \sum_{i=1}^{\infty} \left(\frac{\lambda_i}{1+\lambda_i}\right)^n \lambda_i^{-\sigma} \big(A^\sigma f(t-s), \varphi_i\big) \varphi_i\, ds \right\|^2 \\
&= \sum_{i=1}^{\infty} \left| \sum_{n=N+1}^{\infty} \left(\frac{\lambda_i}{1+\lambda_i}\right)^n \lambda_i^{-\sigma} \int_0^t p_n^{(\alpha)}(s^{1+\alpha}) \big(A^\sigma f(t-s), \varphi_i\big) ds \right|^2 \\
&\leqslant \sum_{i=1}^{\infty} \left[\sum_{n=N+1}^{\infty} \left(\frac{\lambda_i}{1+\lambda_i}\right)^n \lambda_i^{-\sigma} \int_0^t \big|p_n^{(\alpha)}(s^{1+\alpha})\big| \big|\big(A^\sigma f(t-s), \varphi_i\big)\big|\, ds \right]^2.
\end{aligned}$$

As a consequence of Lemma 1 for $\alpha = n$ and $\beta = \sigma$, along with assumption (10), we have

$$\begin{aligned}
\|u(t) - u_N(t)\|^2 &\leqslant C^2 \left(\frac{2\sigma}{e}\right)^{2\sigma} \left[\sum_{n=N+1}^{\infty} \frac{1}{n^{\sigma-\gamma}} \right]^2 \sum_{i=1}^{\infty} \left[\int_0^t \big|\big(A^\sigma f(s), \varphi_i\big)\big|\, ds \right]^2 \\
&\leqslant C^2 \left(\frac{2\sigma}{e}\right)^{2\sigma} \left[\int_N^\infty \frac{dx}{x^{\sigma-\gamma}} \right]^2 \sum_{i=1}^{\infty} t \int_0^t \big|\big(A^\sigma f(s), \varphi_i\big)\big|^2\, ds \\
&= t\, C^2 \frac{(2\sigma)^{2\sigma} e^{-2\sigma}}{(\sigma - \gamma - 1)^2} \frac{1}{N^{2(\sigma-\gamma-1)}} \int_0^t \|A^\sigma f(s)\|^2 ds,
\end{aligned}$$

which gives estimate (16). The theorem is proved. ∎

Combining theorems 1 and 2, we arrive at the main statement of this section.

Theorem 3. *Let the conditions of Theorems 1 and 2 be satisfied. Then the Cayley transform method (8), (9) is a method without saturation of accuracy, and the following error estimate holds true:*

$$\|u(t) - u_N(t)\| \leqslant \frac{M(t)}{N^{\sigma-\gamma-1}} \left\{ \|A^\sigma u_0\| + \left[\int_0^t \|A^\sigma f(s)\|^2 ds \right]^{1/2} \right\},$$

where $M(t) = C \max(1; \sqrt{t}) \frac{(2\sigma)^\sigma e^{-\sigma}}{\sigma-\gamma-1}$ is independent of n.

3. The Cayley transform method with an exponential rate of convergence

At this point, we need the following auxiliary statement.

Lemma 2. *Let $\alpha > 0$. Then the following inequality holds true:*

$$\max_{\lambda > 0} \left(\frac{\lambda}{\lambda+1}\right)^\alpha e^{-\lambda} = e^{\frac{1-\sqrt{4\alpha+1}}{2}} \left(1 - \frac{2}{1 + \sqrt{4\alpha+1}}\right)^\alpha. \tag{17}$$

Proof. We consider the function

$$\varphi(\lambda) = \left(\frac{\lambda}{\lambda+1}\right)^{\alpha} e^{-\lambda}, \quad \lambda > 0.$$

Since $\varphi'(\lambda) = \frac{e^{-\lambda}\lambda^{\alpha-1}(-\lambda^2-\lambda+\alpha)}{(\lambda+1)^{\alpha+1}}$, then we have

$$\max_{\lambda>0}\varphi(\lambda) = \varphi\left(\frac{\sqrt{4\alpha+1}-1}{2}\right) = \left(\frac{\sqrt{4\alpha+1}-1}{\sqrt{4\alpha+1}+1}\right)^{\alpha} e^{-\frac{\sqrt{4\alpha+1}-1}{2}},$$

which leads to inequality (17). The lemma is proved. ∎

We now prove the analogue of Theorem 1.

Theorem 4. *Let $f(t) \equiv 0$, the Laguerre–Cayley polynomials $p_n^{(\alpha)}(x)$ satisfy condition (10), and $u_0 \in D(e^A)$. Then the Cayley transform method (12) is exponentially convergent, and the following error estimate holds true:*

$$\|u(t) - u_N(t)\| \leqslant \sqrt{e}\, C(t) S(\gamma) e^{-\sqrt{N+1}} \|e^A u_0\| \quad (t \in [0; T],\ N \in \mathbb{N}), \tag{18}$$

where $S(\gamma)$ represents the sum of the convergent series: $S(\gamma) = \sum_{n=1}^{\infty} n^{\gamma}\left(1 - \frac{2}{1+\sqrt{4n+1}}\right)^n$.

Proof. From formulas (11) and (12), we can derive the representation

$$\begin{aligned}
\|u(t) - u_N(t)\|^2 &= \left\| \sum_{n=N+1}^{\infty} p_n^{(\alpha)}(t^{1+\alpha})\, Q^n e^{-A} e^A u_0 \right\|^2 \\
&= \left\| \sum_{n=N+1}^{\infty} p_n^{(\alpha)}(t^{1+\alpha}) \sum_{i=1}^{\infty} \left(\frac{\lambda_i}{1+\lambda_i}\right)^n e^{-\lambda_i} (e^A u_0, \varphi_i)\varphi_i \right\|^2 \\
&= \left\| \sum_{i=1}^{\infty} \left[\sum_{n=N+1}^{\infty} p_n^{(\alpha)}(t^{1+\alpha}) \left(\frac{\lambda_i}{1+\lambda_i}\right)^n e^{-\lambda_i}(e^A u_0, \varphi_i) \right] \varphi_i \right\|^2 \\
&= \sum_{i=1}^{\infty} \left| \sum_{n=N+1}^{\infty} p_n^{(\alpha)}(t^{1+\alpha}) \left(\frac{\lambda_i}{1+\lambda_i}\right)^n e^{-\lambda_i}(e^A u_0, \varphi_i) \right|^2 \\
&\leqslant \sum_{i=1}^{\infty} \left|(e^A u_0,\, \varphi_i)\right|^2 \left[\sum_{n=N+1}^{\infty} \left|p_n^{(\alpha)}(t^{1+\alpha})\right| \left(\frac{\lambda_i}{1+\lambda_i}\right)^n e^{-\lambda_i} \right]^2.
\end{aligned}$$

Applying Lemma 2 with $\alpha = n$, we get a chain of inequalities:

$$\begin{aligned}
\|u(t) - u_N(t)\|^2 &\leqslant \left[\sum_{n=N+1}^{\infty} C(t) n^{\gamma} e^{\frac{1-\sqrt{4n+1}}{2}} \left(1 - \frac{2}{1+\sqrt{4n+1}}\right)^n \right]^2 \sum_{i=1}^{\infty} \left|(e^A u_0, \varphi_i)\right|^2 \\
&\leqslant C^2(t) e^{1-\sqrt{4(N+1)+1}} \left[\sum_{n=N+1}^{\infty} n^{\gamma} \left(1 - \frac{2}{1+\sqrt{4n+1}}\right)^n \right]^2 \|e^A u_0\|^2 \\
&= e C^2(t) S^2(\gamma) e^{-\sqrt{4(N+1)+1}} \|e^A u_0\|^2,
\end{aligned}$$

where $S(\gamma) = \sum_{n=1}^{\infty} n^{\gamma}\left(1 - \frac{2}{1+\sqrt{4n+1}}\right)^n$ is the sum of the number series, and its convergence can be shown by the logarithmic criterion:

$$\frac{-\ln\left[n^{\gamma}\left(1 - \frac{2}{1+\sqrt{4n+1}}\right)^n\right]}{\ln n} = -\gamma + \frac{-n\ln\left(1 - \frac{2}{1+\sqrt{4n+1}}\right)}{\ln n} \geqslant l > 1 \quad \forall n \geqslant n_0$$

since

$$\frac{-n\ln\left(1 - \frac{2}{1+\sqrt{4n+1}}\right)}{\ln n} \sim \frac{\frac{2n}{1+\sqrt{4n+1}}}{\ln n} \sim \frac{\sqrt{n}}{\ln n} \to +\infty \quad \text{as} \quad n \to \infty.$$

This immediately leads to estimate (18). The theorem is proved. ∎

Now we establish the analogue of Theorem 2.

Theorem 5. *Let $u_0 = 0$, the Laguerre–Cayley polynomials $p_n^{(\alpha)}(x)$ satisfy inequality (10) with $0 < C(t) \leqslant C$ $\forall t \in [0;T]$, and the vector function $f(t)$ meet the conditions*

$$f(t) \in D(e^A) \quad \forall t \in [0;\, T], \qquad \int_0^t \left\|e^A f(s)\right\|^2 ds < \infty.$$

Then the Cayley transform method (15) is exponentially convergent, and the error estimate holds true:

$$\|u(t) - u_N(t)\| \leqslant \sqrt{e}\sqrt{t}CS(\gamma)e^{-\sqrt{N+1}} \left\{\int_0^t \|e^A f(s)\|^2 ds\right\}^{1/2} \quad (t \in [0;\, T],\ N \in \mathbb{N}) \tag{19}$$

with $S(\gamma)$ described in Theorem 4.

Proof. Using formulas (14) and (15), we obtain the error representation and a number of inequalities:

$$\begin{aligned}
\|u(t) - u_N(t)\|^2 &= \left\|\int_0^t \big(U(s) - U_N(s)\big) f(t-s)\, ds\right\|^2 \\
&= \left\|\int_0^N \sum_{n=N+1}^{\infty} p_n^{(\alpha)}(s^{1+\alpha})\, Q^n e^{-A} e^A f(t-s)\, ds\right\|^2 \\
&= \left\|\int_0^t \sum_{n=N+1}^{\infty} p_n^{(\alpha)}(s^{1+\alpha}) \sum_{i=1}^{\infty} \left(\frac{\lambda_i}{1+\lambda_i}\right)^n e^{-\lambda_i} \big(e^A f(t-s),\, \varphi_i\big)\varphi_i\, ds\right\|^2 \\
&= \sum_{i=1}^{\infty} \left|\sum_{n=N+1}^{\infty} \left(\frac{\lambda_i}{1+\lambda_i}\right)^n e^{-\lambda_i} \int_0^t p_n^{(\alpha)}(s^{1+\alpha}) \big(e^A f(t-s), \varphi_i\big)\, ds\right|^2 \\
&\leqslant \sum_{i=1}^{\infty} \left[\sum_{n=N+1}^{\infty} \left(\frac{\lambda_i}{1+\lambda_i}\right)^n e^{-\lambda_i} \int_0^t |p_n^{(\alpha)}(s^{1+\alpha})|\, \big|\big(e^A f(t-s), \varphi_i\big)\big|\, ds\right]^2.
\end{aligned}$$

Owing to Lemma 2 with $\alpha = n$ and assumption (10), we have

$$\begin{aligned}
\|u(t) - u_N(t)\|^2 &\leqslant \left[\sum_{n=N+1}^{\infty} Cn^\gamma e^{\frac{1-\sqrt{4n+1}}{2}} \left(1 - \frac{2}{1+\sqrt{4n+1}}\right)^n\right]^2 \sum_{i=1}^{\infty} \left[\int_0^t \big|\big(e^A f(t-s),\, \varphi_i\big)\big|\, ds\right]^2 \\
&\leqslant C^2 e^{1-\sqrt{4(N+1)+1}} \left[\sum_{n=N+1}^{\infty} n^\gamma \left(1 - \frac{2}{1+\sqrt{4n+1}}\right)^n\right]^2 \sum_{i=1}^{\infty} t \int_0^t \big|\big(e^A f(t-s), \varphi_i\big)\big|^2 ds \\
&= etC^2 S^2(\gamma) e^{-\sqrt{4(N+1)+1}} \int_0^t \|e^A f(s)\|^2 ds,
\end{aligned}$$

and $S(\gamma)$ has the same meaning as in Theorem 4. This implies estimate (19).

The theorem is proved. ∎

We are now ready to prove the main result of this section, based on Theorems 4 and 5.

Theorem 6. *Let the assumptions of Theorems 4 and 5 be fulfilled. Then the Cayley transform method (8), (9) is exponentially convergent and characterized by the following error estimate:*

$$\|u(t) - u_N(t)\| \leqslant M(t)e^{-\sqrt{N+1}} \left[\|e^A u_0\| + \left\{\int_0^t \|e^A f(s)\|^2 ds\right\}^{1/2}\right] \quad (t \in [0;\, T],\ N \in \mathbb{N}),$$

where $M(t) = \sqrt{e}\, CS(\gamma) \max(1;\, \sqrt{t})$ is independent of n, and $S(\gamma)$ is as introduced in Theorem 4.

4. Numerical examples

We now demonstrate the effectiveness of our approach through several numerical examples.

Example 1. Consider the scalar case of problem (1) for $\alpha = -1/2$, $A = 1 = \text{const}$, $f(t) \equiv 0$, $u_0 = 1$, namely:

$$u'(t) + \frac{d}{dt}\frac{1}{\sqrt{\pi}}\int_0^t \frac{u(s)}{\sqrt{t-s}}ds = 0, \quad t > 0, \quad u(0) = 1.$$

The exact solution $u(t)$ for $t = 2$ can be found by formula (11) (see also (3)):

$$u(2) = E_{1/2}(-\sqrt{2}) = \sum_{j=0}^{\infty} \frac{(-1)^j 2^{j/2}}{\Gamma(1+j/2)} = e^2\big(1 - \operatorname{erf}(\sqrt{2})\big) = 0,33620400244634121\ldots,$$

where $\operatorname{erf}(x) = \frac{2}{\sqrt{\pi}}\int_0^x e^{-t^2}dt$ is the error function. The approximate solution $u_N(t)$ for $t = 2$ can be found by formula (12):

$$u_N(2) = \sum_{n=0}^{N} \frac{1}{2^n} p_n^{(-1/2)}(\sqrt{2}) = 1 + \sum_{n=1}^{N} \frac{1}{2^n} \sum_{r=0}^{n-1} \frac{(-1)^{r+1} C_{n-1}^r 2^{(r+1)/2}}{\Gamma\big(1 + (r+1)/2\big)}.$$

The values of the absolute error $\operatorname{err}_N(2) = \big|u(2) - u_N(2)\big|$ for different N are presented in Table 1.

Table 1. Example 1.

N	$u_N(2)$	$\operatorname{err}_N(2)$
8	0.33617210996594734...	$3.189\ldots \cdot 10^{-5}$
16	0.33620399995084106...	$2.495\ldots \cdot 10^{-9}$
32	0.33620400244634010...	$1.102\ldots \cdot 10^{-15}$
64	0.33620400244634121...	$1.313\ldots \cdot 10^{-28}$

Example 2. Consider the scalar problem (1) for $\alpha = -1/2$, $A = 1 = \text{const}$, $f(t) = e^{-t}\cos(\pi t)$, $u_0 = 0$:

$$u'(t) + \frac{d}{dt}\frac{1}{\sqrt{\pi}}\int_0^t \frac{u(s)}{\sqrt{t-s}}ds = e^{-t}\cos(\pi t), \quad t > 0, \quad u(0) = 0.$$

The exact solution $u(t)$ for $t = 2$ can be found by formula (14):

$$u(2) = \int_0^2 E_{1/2}(-\sqrt{s}) f(2-s) ds = \sum_{j=0}^{\infty} \frac{(-1)^j}{\Gamma\big(1 + j(1+\alpha)\big)} \int_0^2 s^{j/2} e^{-2+s}\cos(\pi s) ds$$

$$= \int_0^2 \big(1 - \operatorname{erf}(\sqrt{s})\big) e^{-2+s}\cos(\pi s) ds = 0.0260732541461872\ldots$$

The approximate solution $u_N(t)$ for $t = 2$ can be found by formula (15):

$$u_N(2) = \int_0^2 U_N(s) f(2-s)\, ds = \sum_{n=0}^{N} \frac{1}{2^n}\int_0^2 p_n^{(-1/2)}(\sqrt{s}) e^{-2+s}\cos(\pi s)\, ds$$

$$= \int_0^2 e^{-2+s}\cos(\pi s) ds + \sum_{n=1}^{N}\frac{1}{2^n}\sum_{r=0}^{n-1}\frac{(-1)^{r+1}C_{n-1}^r}{\Gamma\big(1+(r+1)/2\big)}\int_0^2 s^{(r+1)/2} e^{-2+s}\cos(\pi s)\, ds.$$

The values of the absolute error $\operatorname{err}_N(2) = \big|u(2) - u_N(2)\big|$ for different N are presented in Table 2.

Table 2. Example 2.

N	$u_N(2)$	$\operatorname{err}_N(2)$
8	0.02605879296236312...	$1.446\ldots \cdot 10^{-5}$
16	0.02607325535048029...	$1.204\ldots \cdot 10^{-9}$
32	0.02607325414601831...	$4.068\ldots \cdot 10^{-16}$
64	0.02607325414601872...	$5.520\ldots \cdot 10^{-29}$

Example 3. Consider the scalar case of problem (1) for $\alpha = -1/3$, $A = 1 = \text{const}$, $f(t) \equiv 0$, $u_0 = 1$, namely:

$$u'(t) + \frac{d}{dt}\frac{1}{\Gamma(2/3)}\int_0^t \frac{u(s)}{(t-s)^{1/3}}ds = 0, \quad t > 0, \quad u(0) = 1.$$

The exact solution $u(t)$ for $t = 2$ can be found by formula (11):

$$u(2) = E_{2/3}(-2^{2/3}) = \sum_{j=0}^{\infty} \frac{(-1)^j 2^{2j/3}}{\Gamma\big(1+2j/3\big)} = 0.27587958925617463\ldots$$

The approximate solution $u_N(t)$ for $t = 2$ can be found by formula (12):

$$u_N(2) = \sum_{n=0}^{N} \frac{1}{2^n} p_n^{(-1/3)}(2^{2/3}) = 1 + \sum_{n=1}^{N} \frac{1}{2^n} \sum_{r=0}^{n-1} \frac{(-1)^{r+1} C_{n-1}^{r} 2^{2(r+1)/3}}{\Gamma\big(1+2(r+1)/3\big)}.$$

The values of the absolute error $\mathrm{err}_N(2) = \big|u(2) - u_N(2)\big|$ for different N are presented in Table 3.

Table 3. Example 3.

N	$u_N(2)$	$\mathrm{err}_N(2)$
8	$0.27582422545819167\ldots$	$5.536\ldots \cdot 10^{-5}$
16	$0.27587972571059977\ldots$	$1.364\ldots \cdot 10^{-7}$
32	$0.27587958925635296\ldots$	$1.783\ldots \cdot 10^{-13}$
64	$0.27587958925617463\ldots$	$3.611\ldots \cdot 10^{-25}$

Example 4. Consider the scalar case of problem (1) for $\alpha = -1/3$, $A = 1 = \mathrm{const}$, $f(t) = e^{-t}\cos(\pi t)$, $u_0 = 0$:

$$u'(t) + \frac{d}{dt}\frac{1}{\Gamma(2/3)} \int_0^t \frac{u(s)}{(t-s)^{1/3}} ds = e^{-t}\cos(\pi t), \quad t > 0, \quad u(0) = 0.$$

The exact solution $u(t)$ for $t = 2$ can be found by formula (14):

$$u(2) = \int_0^2 E_{2/3}(-s^{2/3}) f(2-s) ds = \sum_{j=0}^{\infty} \frac{(-1)^j}{\Gamma(1+2j/3)} \int_0^2 s^{2j/3} e^{-2+s}\cos(\pi s) ds = 0.0198545930306002288\ldots$$

The approximate solution $u_N(t)$ for $t = 2$ can be found by formula (15):

$$u_N(2) = \int_0^2 U_N(s) f(2-s) ds = \sum_{n=0}^{N} \frac{1}{2^n} \int_0^2 p_n^{(-1/3)}(s^{2/3}) e^{-2+s}\cos(\pi s) ds$$

$$= \int_0^2 e^{-2+s}\cos(\pi s) ds + \sum_{n=1}^{N} \frac{1}{2^n} \sum_{r=0}^{n-1} \frac{(-1)^{r+1} C_{n-1}^{r}}{\Gamma\big(1+2(r+1)/3\big)} \int_0^2 s^{2(r+1)/3} e^{-2+s}\cos(\pi s) ds.$$

The values of the absolute error $\mathrm{err}_N(2) = \big|u(2) - u_N(2)\big|$ for different N are presented in Table 4.

Table 4. Example 4.

N	$u_N(2)$	$\mathrm{err}_N(2)$
8	$0.019831620240685168\ldots$	$2.297\ldots \cdot 10^{-5}$
16	$0.019854640847382624\ldots$	$4.781\ldots \cdot 10^{-8}$
32	$0.019854593030651958\ldots$	$5.172\ldots \cdot 10^{-14}$
64	$0.019854593030600228\ldots$	$2.843\ldots \cdot 10^{-25}$

The obtained numerical results demonstrate the exponential rate of convergence of the Cayley transform method for the studied cases involving analytic input data.

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Точність методу перетворення Келі для еволюційного рівняння з дробовою похідною

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Досліджено задачу Коші для диференціального рівняння з похідною дробового порядку і додатно визначеним операторним коефіцієнтом у гільбертовому просторі. Точний розв'язок зображено за допомогою розв'язувального оператора (поданого через функцію Міттаг–Леффлера у вигляді ряду за степенями перетворення Келі оператора та поліномів Лагерра–Келі), а також згортки розв'язувального оператора з правою частиною рівняння. Наближений розв'язок є скінченною сумою перших N доданків цього ряду. Одержано оцінки похибки через параметр дискретизації N за різних припущень щодо гладкості вхідних даних, а саме: метод має степеневу швидкість збіжності та властивість ненасичення точності у випадку скінченної диференційовності правої частини рівняння; метод є експоненціально збіжним, якщо права частина є аналітичною (в певному сенсі) функцією.

Ключові слова: *задача Коші; гільбертів простір; функція Міттаг–Леффлера; перетворення Келі; метод без насичення точності; експоненціально збіжний алгоритм.*