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SENSITIVITY ANALYSIS OF CONTROL SYSTEMS SYNTHESIZED BY FEEDBACK CONTROL METHODS TO CHANGES IN THE MOMENT OF INERTIA OF THE SECOND MASS OF A TWO-MASS POSITIONING SYSTEM

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Abstract: In this work, a comprehensive approach to the sensitivity analysis of state-variable control systems is proposed. The advantages of systems synthesized by a feedback linearization method are demonstrated being compared to a system synthesized by the modal control method, both in terms of sensitivity to changes in the moment of inertia of the second mass and in terms of control quality. The influence of applying a PI controller and a PI$^{\mu}$-controller on the sensitivity of the system to changes in the moment of inertia of the second mass and on the speed of response and overshoot of the output coordinate in systems synthesized by the feedback linearization method is analyzed.

Keywords: feedback linearization method, PI controller, sensitivity function of the synthesized system, two-mass system.

Introduction

In many modern technological processes, in particular, the manufacture of metal products on rolling mills, the paper industry, long shafts are available to connect an electromechanical converter with an actuator. The consideration of processes in such electric drives is traditionally reduced to the analysis of work based on a two-mass model (Fig. 1) [1–3].

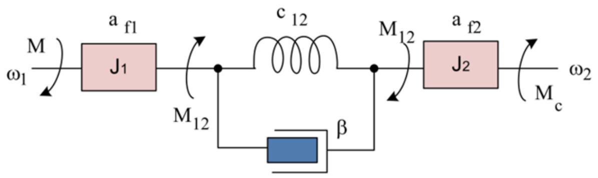


Fig. 1. Structural diagram of the two-mass system model

To reduce the order of the system under study when analyzing processes in the two-mass system and the actuator on the basis of the theory of different-speed systems, processes in an electromagnetic circuit are neglected. In addition to speed control systems, in technological processes there is often a need to regulate the position. Such problems are characteristic, in particular, for robotic systems [4], or steel smelting systems in arc steelmaking furnaces [5]. In these cases, the two-mass model is also used to consider the processes. The mathematical description of the two-mass positional system shown in Fig. 1 in a vector-matrix form has the form:

$$
\frac{d}{dt}\underbrace{\begin{pmatrix}\omega_1\\ M_{12}\\ \omega_2\\ \varphi\end{pmatrix}}_{X} =
$$

$$
= \underbrace{\begin{pmatrix}\frac{-\beta-a_{f1}}{J_1} & \frac{-1}{J_1} & \frac{\beta}{J_1} & 0\\ c_{12} & 0 & -c_{12} & 0\\ \frac{\beta}{J_2} & \frac{1}{J_2} & \frac{-\beta-a_{f2}}{J_2} & 0\\ 0 & 0 & 1 & 0\end{pmatrix}}_{A} \cdot \underbrace{\begin{pmatrix}\omega_1\\ M_{12}\\ \omega_2\\ \varphi\end{pmatrix}}_{X} + \underbrace{\begin{pmatrix}\frac{1}{J_1}\\ 0\\ 0\\ 0\end{pmatrix}}_{B} \cdot M + \underbrace{\begin{pmatrix}0\\ 0\\ \frac{-1}{J_2}\\ 0\end{pmatrix}}_{B1} \cdot M_c
$$

$$
Y = \underbrace{\begin{bmatrix}0 & 0 & 0 & 1\end{bmatrix}}_{C_{fIPI}} \cdot x
$$

where A, B, B1 are the matrix of the system and the matrices of the control and disturbing influences, respectively; C is the observability matrix of the system; M is the moment on the engine shaft; ω_1 and ω_2 are the angular velocities of rotation of the first and second masses of the two-mass system; M_{12} is the elastic moment; φ is the displacement; a_{f1}, a_{f2}, β are coefficients characterizing the influence of external and internal viscous friction; J_1, J_2 are moments of inertia of the first and second masses; c_{12} is the shaft stiffness coefficient.

Such a two-mass system is characterized by the presence of natural shaft oscillations with a frequency $f = \frac{1}{2\cdot\pi}\sqrt{\frac{c_{12}\cdot(J_1+J_2)}{J_1\cdot J_2}}$ [1, 2, 6], as well as changes in parameters during the system operation. Characteristic for technological processes is the change in the moment of inertia of the second mass J_2 [7]. The works [8–10] are devoted to reducing the influence of oscillatory processes in the two-mass systems. A typical approach to the synthesis of control systems in the two-mass systems is controlling by the state variables of the system [9, 11–13]. In this case, such tasks as the synthesis of estimators [11, 12] to obtain information about the state variables of the system and the synthesis of coupling coefficients by the state variables, which will ensure robustness to parametric disturbances with high control quality [13, 14], are solved.

Traditionally, to reduce the sensitivity to parametric disturbances, sliding control or robust control based on H_2 , H_∞ norms is used [13–17]. An important element of the synthesis of feedback coefficients for the state variables of the system is the placement of the poles of the system transfer function [18, 19]. In the case of the synthesis of modal control for the full state vector [5] or control using the feedback linearization method [20, 21], the feedback coefficients for the state variables are determined on the basis of one of the standard linear forms. In this case, the task of choosing the geometric mean root ω_0 that determines the cutoff frequency of the system and affects the value of the feedback coefficients for the state variables arises. In the case of positional systems, the synthesis of control effects using the two-mass model of the mechanical part [11, 22, 23] may be complicated by additional restrictions on the accuracy of working out a given trajectory of motion or positioning of the actuator, the absence of overshoot, etc.

Considering the widespread use of control systems based on the variables of the system state vector and the presence of parametric disturbances caused by both the peculiarities of the technological process and the errors in determining the parameters of the two-mass model of the mechanical part, it is advisable, in our opinion, to analyze the parametric sensitivity of various control systems.

Sensitivity analysis of a system synthesized by the modal control method.

The matrix transfer function of a closed-loop system in the case of full state vector control is defined as follows:

$$\boldsymbol{T}(s)=\left(s\cdot\mathbf{I}-\mathbf{A}+\mathbf{B}\cdot\mathbf{K}\right)^{-1}\cdot\mathbf{B},$$

where s is the Laplace operator; $\mathbf{I}$ is the identity matrix; $\mathbf{K}=\begin{bmatrix}k_1 & k_2 & k_3 & k_4\end{bmatrix}$ is the vector of feedback coefficients on the state variables of the system. Then the transfer function from the initial coordinate of the system to the control signal (here and further the influence of external viscous friction $a_{f1}=0$ $a_{f2}=0$ is neglected) is as follows:

$$\frac{\varphi(s)}{M(s)}=W(s)=\frac{Num(s)}{Den(s)}=\frac{\dfrac{c_{12}+\beta\cdot s}{J_1\cdot J_2}}{s^4+\dfrac{\beta\cdot(J_1+J_2)+J_2\cdot k_1}{J_1\cdot J_2}\cdot s^3+\dfrac{c_{12}\cdot(J_1+J_2+J_2\cdot k_2)+\beta\cdot(k_1+k_3)}{J_1\cdot J_2}\cdot s^2+}$$

$$\to\frac{c_{12}\cdot(k_1+k_3)+\beta\cdot k_4}{J_1\cdot J_2}\cdot s+\frac{c_{12}\cdot k_4}{J_1\cdot J_2}$$

Accordingly, with the desired location of the roots of the characteristic polynomial $H(s)=s^4+\alpha_1\cdot\omega_0\cdot s^3+$ $+\alpha_2\cdot\omega_0^2\cdot s^2+\alpha_3\cdot\omega_0^3\cdot\cdot s+\alpha_4\cdot\omega_0^4$ given, the vector of feedback coefficients on the state variables as a solution to the system of equations formed by equating the coefficients at the corresponding powers of the polynomials *Den(s)* and *H(s)* is found. In the case of being focused on the binomial form of the root distribution, the feedback coefficients on the state variables of the system are respectively equal to:

$$k_1=\frac{J_1\cdot J_2\cdot 4\cdot\omega_0-\beta\cdot(J_1+J_2)}{J_2};\ k_4=\frac{\omega_0^4\cdot J_1\cdot J_2}{c_{12}}$$

$$k_2=\frac{J_1\cdot J_2\cdot 6\cdot\omega_0^2-c_{12}\cdot(J_1+J_2)}{J_2\cdot c_{12}}-$$
$$-\frac{4\cdot\omega_0^3\cdot J_1\cdot\beta}{c_{12}^2}+\frac{\omega_0^4\cdot J_1\cdot\beta^2}{c_{12}^3}$$

$$k_3=\frac{\beta\cdot(J_1+J_2)-J_1\cdot J_2\cdot 4\cdot\omega_0}{J_2}+\frac{4\cdot\omega_0^3\cdot J_1\cdot J_2}{c_{12}}-$$
$$-\frac{\omega_0^4\cdot J_1\cdot J_2\cdot\beta}{c_{12}^2}$$

On the other hand, in steady state at s = 0:

$$\frac{\varphi(0)}{M(0)}=\frac{\dfrac{c_{12}}{J_1\cdot J_2}}{\dfrac{c_{12}\cdot k_4}{J_1\cdot J_2}}=\frac{1}{k_4}$$

and therefore:

$$\frac{\omega_0^4\cdot J_1\cdot J_2}{c_{12}}=\frac{M(t\to\infty)}{\varphi(t\to\infty)}$$

and then:

$$\omega_0^4=\frac{M(t\to\infty)}{\varphi(t\to\infty)}\cdot\frac{c_{12}}{J_1\cdot J_2}\cdot\frac{J_1+J_2}{J_1+J_2}=$$
$$=\frac{M(t\to\infty)}{\varphi(t\to\infty)}\cdot\frac{(2\cdot\pi\cdot f)^2}{J_1+J_2},$$

where f is the natural frequency of oscillations of the long shaft.

In the case of changing the moment of inertia of the second mass ΔJ_2 to matrix A of the system, it will take the following form:

$$A^* = \begin{pmatrix} \dfrac{-\beta - a_{f1}}{J_1} & \dfrac{-1}{J_1} & \dfrac{\beta}{J_1} & 0 \\ c_{12} & 0 & -c_{12} & 0 \\ \dfrac{\beta}{J_2 + \Delta J_2} & \dfrac{1}{J_2 + \Delta J_2} & \dfrac{-\beta - a_{f2}}{J_2 + \Delta J_2} & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

The matrix transfer function of the closed-loop system is determined taking into account the change in the moment of inertia of the second mass:

$$\boldsymbol{T}(s) = (s \cdot \boldsymbol{I} - \boldsymbol{A}^* + \boldsymbol{B} \cdot \boldsymbol{K})^{-1} \cdot \mathbf{B},$$

and the transfer function from the system initial coordinate to the control signal is as follows:

$$\frac{\varphi(s)}{M(s)} = W^*(s) = \frac{Num^*(s)}{Den^*(s)} =$$

$$= \frac{\dfrac{c_{12} + \beta \cdot s}{J_1 \cdot J_2}}{\dfrac{\Delta S(s)}{J_1 \cdot J_2} + s^4 + \dfrac{\beta \cdot (J_1 + J_2) + J_2 \cdot k_1}{J_1 \cdot J_2} \cdot s^3 + \dfrac{c_{12} \cdot (J_1 + J_2 + J_2 \cdot k_2) + \beta \cdot (k_1 + k_3)}{J_1 \cdot J_2} \cdot s^2 + \dfrac{c_{12} \cdot (k_1 + k_3) + \beta \cdot k_4}{J_1 \cdot J_2} \cdot s + \dfrac{c_{12} \cdot k_4}{J_1 \cdot J_2}}$$

where

$$S(s) = \Delta J_2 \cdot \left(J_1 \cdot s^4 + (\beta + k_1) \cdot s^3 + c_{12} \cdot (1 + k_2) \cdot s^2\right).$$

To analyze the sensitivity of the system to the action of parametric disturbances, either the distance between the transfer functions is used [24]:

$$\Psi(W, W^*) = \frac{|W(j\omega) - W^*(j\omega)|}{\sqrt{1 + |W(j\omega)|^2} \cdot \sqrt{1 + |WW^*(j\omega)(j\omega)|^2}}$$

or the sensitivity function is applied [25]:

$$Sens(s, \gamma) = \frac{\dfrac{W(s,\gamma) - W^*(s, \gamma + \Delta\gamma)}{W(s,\gamma)}}{\dfrac{\Delta\gamma}{\gamma}}.$$

Having constructed the logarithmic amplitude-phase frequency characteristic (LPFC) of the sensitivity function and analyzing the value of the system gain to the cutoff frequency, we can conclude about the sensitivity of the synthesized system to parametric perturbation.

The sensitivity function to changes in the moment of inertia of the second mass taking into account:

$$Den^*(s) = Den(s) + \frac{\Delta S(s)}{J_1 \cdot J_2}, \qquad Num^*(s) = Num(s)$$

will take the following form:

$$Sens(s, J_2) = \frac{\dfrac{\dfrac{Num(s)}{Den(s)} - \dfrac{Num^*(s)}{Den^*(s)}}{\dfrac{Num(s)}{Den(s)}}}{\dfrac{\Delta J_2}{J_2}} =$$

$$= \frac{J_2 \, g \Delta S(s)}{\Delta J_2 \, g\left(J_1 \, g J_2 \, g Den(s) + \Delta\, S(s)\right)}$$

and taking into account the synthesized feedback coefficients on the system state variables:

$$Sens(s, J_2) = \frac{J_2 \cdot \left(\begin{array}{c} J_1 \cdot s^4 + \left(\beta + \dfrac{J_1 \cdot J_2 \cdot 4 \cdot \omega_0 - \beta \cdot (J_1 + J_2)}{J_2}\right) \cdot s^3 + \rightarrow \\ \rightarrow c_{12} \cdot \left(1 + \dfrac{J_1 \cdot J_2 \cdot 6 \cdot \omega_0^2 - c_{12} \cdot (J_1 + J_2)}{J_2 \cdot c_{12}} - \dfrac{4 \cdot \omega_0^3 \cdot J_1 \cdot \beta}{c_{12}^2} + \dfrac{\omega_0^4 \cdot J_1 \cdot \beta^2}{c_{12}^3}\right) \cdot s^2 \end{array} \right)}{\begin{array}{c} J_1 \cdot J_2 \cdot (s + \omega_0)^4 + \Delta J_2 \cdot \left(J_1 \cdot s^4 + \left(\beta + \dfrac{J_1 \cdot J_2 \cdot 4 \cdot \omega_0 - \beta \cdot (J_1 + J_2)}{J_2}\right) \cdot s^3\right) + \rightarrow \\ \rightarrow \Delta J_2 \cdot c_{12} \cdot \left(1 + \dfrac{J_1 \cdot J_2 \cdot 6 \cdot \omega_0^2 - c_{12} \cdot (J_1 + J_2)}{J_2 \cdot c_{12}} - \dfrac{4 \cdot \omega_0^3 \cdot J_1 \cdot \beta}{c_{12}^2} + \dfrac{\omega_0^4 \cdot J_1 \cdot \beta^2}{c_{12}^3}\right) \cdot s^2 \end{array}}.$$

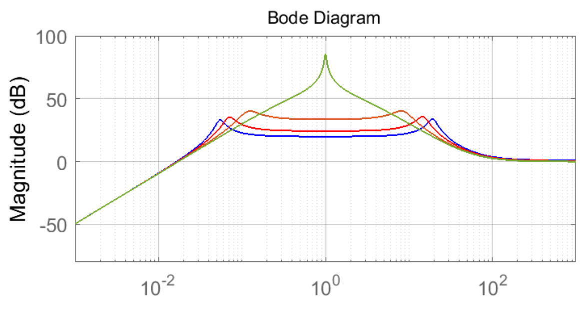


Fig. 2. LPFC of the sensitivity function of the control system for the full state vector when the moment of inertia of the second mass is decreasing (—) and increasing (__)

Fig. 2 shows the LPFC of the sensitivity functions according to the full vector of the state of the control system synthesized for the binomial form of the root distribution at $\omega_0 = 1$, obtained for the following system parameters: moments of inertia of the first and second masses $J_1 = 1$ кг $\cdot$ м^2, $J_2 = 1$ кг $\cdot$ м^2; coefficient of internal viscous friction $\omega_0 = 1, J_1 = 1\ kg \cdot m^2$, $J_2 = 1\ kg \cdot m^2$, $\beta = 10$; shaft stiffness coefficient $c_{12} = 10000$. The obtained results demonstrate significant sensitivity of the system to changes in the moment of inertia of the second mass, especially in the direction of its increase.

The transfer function of the system when the moment of inertia of the second mass is being changed and after substituting the synthesized feedback coefficients for the system state variables and replacing: $\Delta J_2 = \delta \cdot J_2$ will have the form

$$W^*(s) = \frac{\varphi(s)}{M(s)} = \frac{\frac{c_{12}+\beta\cdot s}{J_1\cdot J_2\cdot(1+\delta)}}{s^4+\left(4\cdot\omega_0-\frac{J_1}{J_2}\cdot\frac{\delta}{1+\delta}\cdot\beta\right)\cdot s^3+4\cdot\omega_0^3\cdot\frac{1}{1+\delta}\cdot s+\omega_0^4\cdot\frac{1}{1+\delta}+\rightarrow \atop \rightarrow\left(6\cdot\omega_0^2-\frac{J_1}{J_2}\cdot\frac{\delta}{1+\delta}\cdot c_{12}-4\cdot\omega_0^3\cdot\frac{\delta}{1+\delta}\cdot\frac{\beta}{c_{12}}+\frac{\delta}{1+\delta}\cdot\frac{\beta^2}{c_{12}^2}\cdot\omega_0^4\cdot\right)\cdot s^2}.$$

Analysis of the obtained transfer function allows for concluding that at certain ratios of the system parameters, the change in the moment of inertia of the second mass can cause instability of the positional system. The range of change in the deviations $[\delta_{\min}, \delta_{\max}]$ of the moment of inertia of the second mass, as a function of the system parameters, can be easily found using the algebraic Hurwitz criterion. To obtain the range of change, it is enough to find the second-order determinant of the Hurwitz matrix and, equating it to zero, find the roots of the quadratic equation with respect to δ:

$$det\begin{pmatrix} 4\cdot\omega_0-\frac{J_1}{J_2}\cdot\frac{\delta}{1+\delta}\cdot\beta & 4\cdot\omega_0^3\cdot\frac{1}{1+\delta} \\ 1 & 6\cdot\omega_0^2-\frac{J_1}{J_2}\cdot\frac{\delta}{1+\delta}\cdot c_{12}-4\cdot\omega_0^3\cdot\frac{\delta}{1+\delta}\cdot\frac{\beta}{c_{12}}+\frac{\delta}{1+\delta}\cdot\frac{\beta^2}{c_{12}^2}\cdot\omega_0^4 \end{pmatrix} = 0.$$

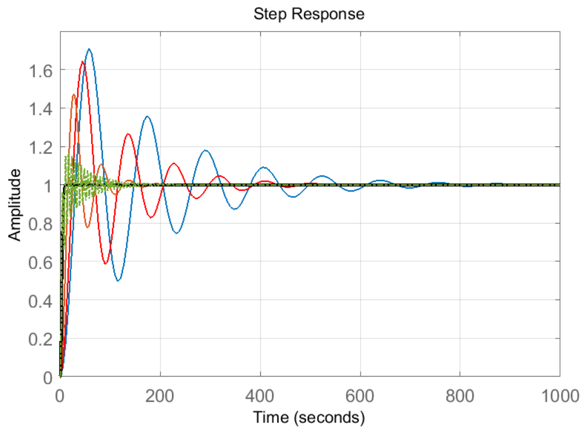


Fig. 3. Transient characteristics of the positional modal control system at a value corresponding to the parameters at the stage of system synthesis (___), as well as when the moment of inertia of the second mass is decreasing (__ 10 % ___ 6 % ___2 %) and increasing (_ _0.1 %)

Fig. 3 shows the transient characteristics of a two-mass positional system when the moment of inertia of the second mass and the system parameters given above change. Analysis of the obtained results makes it possible to state that a decrease in the moment of inertia of the second mass in a positional system synthesized by the modal control method leads to an increase in the oscillations of the system. At the same time, even a slight increase in this parameter can lead to unstable operation of such a system.

Thus, the application of the modal control method to obtain the desired nature of the transient process in a positional two-mass system requires fairly precise models of the system.

Sensitivity analysis of a system synthesized by the feedback linearization method.

With the observation matrix $C = [0 \quad 0 \quad 0 \quad 1]$, the matrix transfer function of a closed-loop system synthesized by the feedback linearization method is defined as follows:

$$\boldsymbol{T_{fl}}(s) =$$
$$= \left(s\cdot\mathbf{I}-\mathbf{A}+\mathbf{B}\cdot(\mathbf{C}\cdot\mathbf{A}^2\cdot\mathbf{B})^{-1}(\mathbf{C}\cdot\mathbf{A}^3+\mathbf{K}_{fl}\cdot\mathbf{Z})\right)^{-1}\times$$
$$\times\,\mathbf{B}\cdot(\mathbf{C}\cdot\mathbf{A}^2\cdot\mathbf{B})^{-1},$$

where s is the Laplace operator; I is the identity matrix; $\mathbf{K}_{fl} = [k_{1fl} \quad k_{2fl} \quad k_{3fl}]$ is the vector of feedback coefficients in the variables $y, \dot{y}, \ddot{y}$; $\mathbf{Z} = \begin{bmatrix} C \\ C\cdot A \\ C\cdot A^2 \end{bmatrix}$ is the matrix of the connection between the variables $y, \dot{y}, \ddot{y}$ and the state vector of the system x. The conversion function from the initial coordinate of the system to the control signal will have the form:

$$W_{fl}(s) = \frac{1}{s^3+k_{3fl}\cdot s^2+k_{2fl}\cdot s+k_{1fl}}.$$

Equating the coefficients of the denominator of the transfer function of the system to the coefficients at the corresponding powers of the characteristic polynomial, which provides the desired placement of the roots, we obtain $W_{fl}(s)H_{afl}(s) = s^3+\alpha_{1fl}\cdot\omega_0\cdot s^2+$
$+\alpha_{2fl}\cdot\omega_0^2\cdot s+\alpha_{3fl}\cdot\omega_0^3 k_{3fl} = \alpha_{1fl}\cdot\omega_0,$
$k_{2fl} = \alpha_{2fl}\cdot\omega_0^2 k_{1fl} = \alpha_{3fl}\cdot\omega_0^3$. Respectively, the matrix transfer function, which takes into account the change in the moment of inertia of the second mass, will have the form:

$$\mathbf{T}_{fl}^*(s) = \left(s\cdot\boldsymbol{I}-\boldsymbol{A}^*+\boldsymbol{B}\cdot(\boldsymbol{C}\cdot\boldsymbol{A}^2\cdot\boldsymbol{B})^{-1}(\boldsymbol{C}\cdot\boldsymbol{A}^3+\right.$$
$$\left.+\boldsymbol{K}_{fl}\cdot\boldsymbol{Z}^*)\right)^{-1}\cdot\mathbf{B}\cdot(\mathbf{C}\cdot\mathbf{A}^2\cdot\mathbf{B})^{-1}.$$

Here

$$\boldsymbol{Z}^* = \begin{bmatrix} C \\ C \cdot \boldsymbol{A}^* \\ C \cdot \boldsymbol{A}^{*2} \end{bmatrix}$$

is the matrix of the connection between the variables $y, \dot{y}, \ddot{y}$ and the state vector of the system x.

And the transfer function from the initial system coordinate to the control signal is as follows:

$$W^*{}_{fl}(s) = \frac{1}{s^3+k_{3fl}\cdot s^2+k_{2fl}\cdot s+k_{1fl}+\frac{\Delta J_2}{J_2}\cdot\left(s^3-\frac{\beta}{J_2}\cdot s^2\right)}.$$

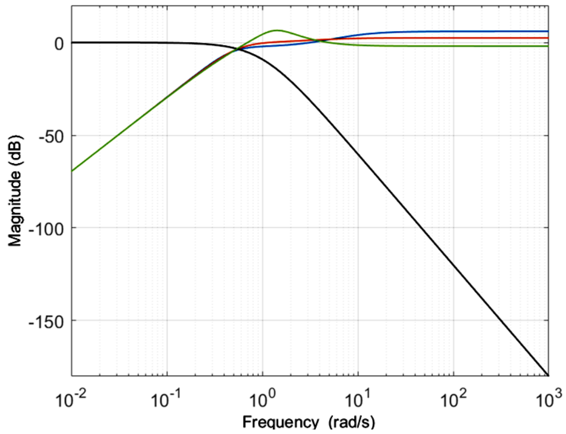
Then, taking into account the synthesized feedback coefficients, the sensitivity function will take the form:

$$Sens_{fl}(s,J_2) = \frac{s^3-\frac{\beta}{J_2}\cdot s^2}{(s+\omega_0)^3+\frac{\Delta J_2}{J_2}\cdot\left(s^3-\frac{\beta}{J_2}\cdot s^2\right)}.$$

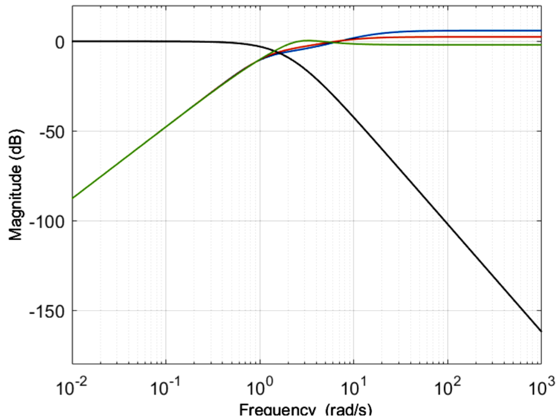
Fig. 4 shows the LPFC of the sensitivity function of the control system synthesized by the feedback linearization method when tuned to the binomial form of the root distribution at $\omega_0 = 1$, and the LPFC of such a system. The results obtained demonstrate that the synthesized system is sensitive to both a decrease and an increase in the moment of inertia of the second mass. Moreover, the sensitivity to changes in the moment of inertia of the second mass ΔJ_2 increases with a decrease in the value of J_2. At the same time, the sensitivity of the system decreases with an increase in the value of the geometric mean root ω_0.

The transfer function of the system after substituting the values of the synthesized coefficients of the control system will have the form:

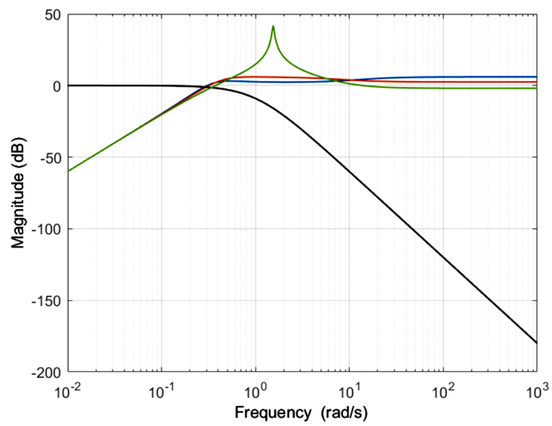
$$W^*_{fl}(s) = \frac{\frac{1}{1+\delta}}{s^3+\left(3\cdot\omega_0\cdot\frac{1}{1+\delta}-\frac{\beta}{J_2}\cdot\frac{\delta}{1+\delta}\right)\cdot s^2+3\cdot\omega_0^2\cdot\frac{1}{1+\delta}\cdot s+\omega_0^3\cdot\frac{1}{1+\delta}}.$$



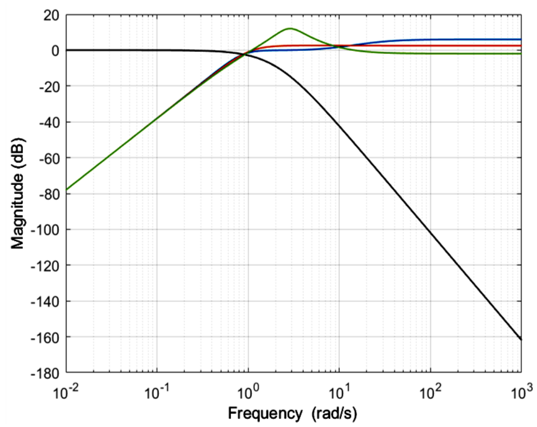
a) J1=1; J2=3; c12=10000;β=10;ω₀=1



c) J1=1; J2=3; c12=10000;β=10;ω₀=2



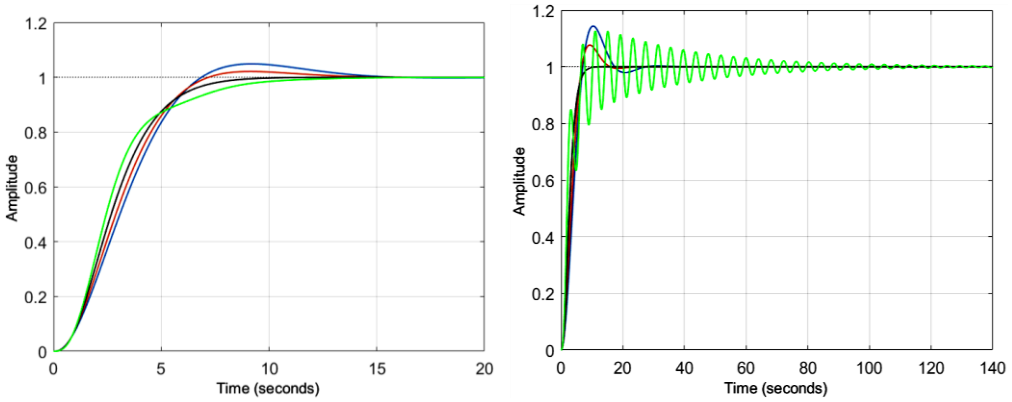
b) J1=1; J2=1; c12=10000;β=10;ω₀=1



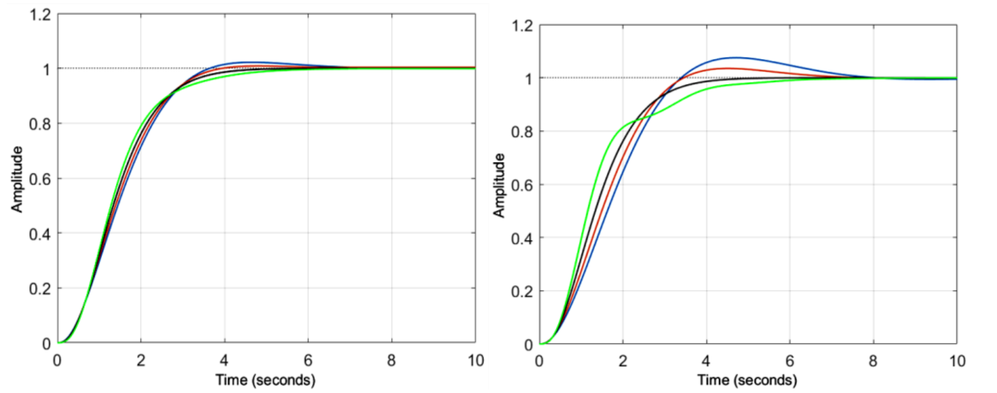
d) J1=1; J2=1; c12=10000;β=10;ω₀=2

Fig. 4. LPFC of the system synthesized by the feedback linearization method (__) and the sensitivity function of the system when decreasing (__50 %, __ 25 %) and increasing (__25 %) moment of inertia of the second mass

Fig. 5 shows the transient responses of a two-mass positioning system synthesized using the feedback linearization method under variations in the moment of inertia of the second mass. The obtained dependencies confirm the conclusion that such a system has significantly lower sensitivity to changes in the studied parameter, compared to the system synthesized by the modal control method. As in the previous system, an increase in the moment of inertia relative to the value used for synthesizing the control system parameters leads to instability. At the same time, a lower moment of inertia compared to the synthesis value increases overshoot and reduces the response speed of the system. With an increase in the value of the geometric mean root ω_0, the overshoot in the system decreases.



a) J_1=1; J_2=3; c_{12}=10000; β=10; ω_0=1 *b) J_1=1; J_2=1; c_{12}=10000; β=10; ω_0=1*



c) J1=1; J2=3; c12=10000;β=10;ω_0=2 *d) J1=1; J2=1; c12=10000;β=10;ω_0=2*

Fig. 5. Transient characteristics of the positional control system synthesized by the feedback linearization method at a value corresponding to the parameters at the stage of system synthesis (___), as well as when decreasing (__50 %, __ 25 %) and increasing (__25 %) the moment of inertia of the second mass

Sensitivity analysis of a system synthesized using the feedback linearization method and the use of a PI controller.

In the case of a control system synthesized by the feedback linearization method with the use of a PI controller [21], a variable representing the integral of the position coordinate is applied in forming the feedbacks of the system. To implement this control action, a fictitious variable θ is introduced into the system model. Then the model of such a system will take the form:

$$\underbrace{\frac{d}{dt}\begin{pmatrix}\omega_1\\ M_{12}\\ \omega_2\\ \varphi\\ \theta\end{pmatrix}}_{x1} = \underbrace{\begin{pmatrix}\frac{-\beta - a_{f1}}{J_1} & \frac{-1}{J_1} & \frac{\beta}{J_1} & 0 & 0\\ c_{12} & 0 & -c_{12} & 0 & 0\\ \frac{\beta}{J_2} & \frac{1}{J_2} & \frac{-\beta - a_{f2}}{J_2} & 0 & 0\\ 0 & 0 & 1 & 0 & 0\\ 0 & 0 & 0 & 1 & 0\end{pmatrix}}_{A_{flPI}} \cdot \underbrace{\begin{pmatrix}\omega_1\\ M_{12}\\ \omega_2\\ \varphi\\ \theta\end{pmatrix}}_{x1} + \underbrace{\begin{pmatrix}\frac{1}{J_1}\\ 0\\ 0\\ 0\\ 0\end{pmatrix}}_{B_{flPI}} \cdot M + \underbrace{\begin{pmatrix}0\\ 0\\ \frac{-1}{J_2}\\ 0\\ 0\end{pmatrix}}_{B1_{flPI}} \cdot M_c$$

$$Y = \underbrace{[\mathbf{0}\quad \mathbf{0}\quad \mathbf{0}\quad \mathbf{1}\quad \mathbf{0}]}_{C_{flPI}} \cdot x1$$

The matrix transfer function of a closed-loop system synthesized by the feedback linearization method and using the PI controller is defined as follows:

$$\boldsymbol{T_{flPI}}(s) =$$

$$= (s \cdot \boldsymbol{I} - \boldsymbol{A_{flPI}} + \boldsymbol{B_{flPI}} \cdot (\boldsymbol{C_{flPI}} \cdot \boldsymbol{A^2_{flPI}} \cdot \boldsymbol{B_{flPI}})^{-1} \times$$
$$\times \ (\boldsymbol{C_{flPI}} \cdot \boldsymbol{A^3_{flPI}} + k_p \cdot \boldsymbol{K_{flPI}} \cdot \boldsymbol{Z_{flPI}} + k_I \cdot \boldsymbol{K_{flPI}} \cdot \boldsymbol{Z1_{flPI}})) \times$$
$$\times \boldsymbol{B_{flPI}} \cdot (\boldsymbol{C_{flPI}} \cdot \boldsymbol{A^2_{flPI}} \cdot \boldsymbol{B_{flPI}})^{-1},$$

where s is the Laplace operator; I5x5 is the identity matrix; $\boldsymbol{K_{flPI}} = [k_{1flr} \quad k_{2flr} \quad k_{3flr}]$ is the vector of feedback coefficients for the variables $y, \dot{y}, \ddot{y}$;

$$\boldsymbol{Z_{flPI}} = \begin{bmatrix} \boldsymbol{C_{flPI}} \\ \boldsymbol{C_{flPI}} \cdot \boldsymbol{A_{flPI}} \\ \boldsymbol{C_{flPI}} \cdot \boldsymbol{A^2_{flPI}} \end{bmatrix}$$

is the matrix of the connection between the variables $y, \dot{y}, \ddot{y}$ and the state vector of the system *x*;

$\boldsymbol{C1_{flPI}} = [\boldsymbol{0} \quad \boldsymbol{0} \quad \boldsymbol{0} \quad \boldsymbol{0} \quad \boldsymbol{1}]$ is the matrix of the connection between the variables $\int ydt, y, \dot{y}$ and the state vector of the system *x*.

The transfer function from the system initial coordinate to the control signal will have the form:

$$W_{fl-PI}(s) = \frac{k_i + k_p \cdot s}{s^4 + k_p \cdot k_{3flr} \cdot s^3 + \left(k_i \cdot k_{3flr} + k_p \cdot k_{2flr}\right) \cdot s^2 + \left(k_i \cdot k_{2flr} + k_p \cdot k_{1flr}\right) \cdot s + k_i \cdot k_{1flr}}.$$

The synthesis of the control system coefficients, which provides a binomial form of the distribution of the roots of the characteristic polynomial, is shown in [32].

Accordingly, the matrix transfer function, which takes into account the change in the moment of inertia of the second mass, will have the form:

$$\boldsymbol{T^*_{flPI}}_{flPI}(s) =$$

$$= (s \cdot \boldsymbol{I} - \boldsymbol{A^*_{flPI}} + \boldsymbol{B_{flPI}} \cdot (\boldsymbol{C_{flPI}} \cdot \boldsymbol{A^2_{flPI}} \cdot \boldsymbol{B_{flPI}})^{-1} \times$$
$$\times (\boldsymbol{C_{flPI}} \cdot \boldsymbol{A^3_{flPI}} + k_p \cdot \boldsymbol{K_{flPI}} \cdot \boldsymbol{Z^*_{flPI}} + k_I \cdot \boldsymbol{K_{flPI}} \times$$
$$\times \boldsymbol{Z1^*_{flPI}})) \cdot \boldsymbol{B_{flPI}} \times$$
$$\times (\boldsymbol{C_{flPI}} \cdot \boldsymbol{A^2_{flPI}} \cdot \boldsymbol{B_{flPI}})^{-1}$$

where the matrix of the system $\boldsymbol{A^*_{flPI}}$ differs from the matrix $\boldsymbol{A_{flPI}}$ by changing the moment of inertia of the second mass ΔJ_2, $\boldsymbol{Z^*_{flPI}} = \begin{bmatrix} \boldsymbol{C_{flPI}} \\ \boldsymbol{C_{flPI}} \cdot \boldsymbol{A^*_{flPI}} \\ \boldsymbol{C_{flPI}} \cdot \boldsymbol{A^{*2}_{flPI}} \end{bmatrix}$ is the matrix of the connection between the variables $y, \dot{y}, \ddot{y}$ and the state vector of the system x; $\boldsymbol{Z1^*_{flPI}} = \begin{bmatrix} \boldsymbol{C1_{flPI}} \\ \boldsymbol{C1_{flPI}} \cdot \boldsymbol{A^*_{flPI}} \\ \boldsymbol{C1_{flPI}} \cdot \boldsymbol{A^{*2}_{flPI}} \end{bmatrix}$ is the matrix of the connection between the variables $\int ydt, y, \dot{y}$ and the state vector of the system x. At the same time, the transfer function will take the form:

$$W^*_{fl-PI}(s) = \frac{k_i + k_p \cdot s}{s^4 + k_p \cdot k_{3flr} \cdot s^3 + \left(k_i \cdot k_{3flr} + k_p \cdot k_{2flr}\right) \cdot s^2 + \left(k_i \cdot k_{2flr} + k_p \cdot k_{1flr}\right) \cdot s + \rightarrow \\ \rightarrow k_i \cdot k_{1flr} + \frac{\Delta J_2}{J_2} \cdot \left(s^4 - \frac{\beta}{J_2} \cdot s^3\right)}$$

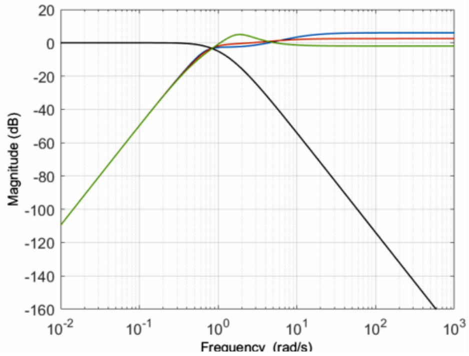
Then, taking into account the synthesized feedback coefficients and PI controller coefficients, the sensitivity function will be as follows:

$$Sens_{fl-PI}(s, J_2) = \frac{s^4 - \frac{\beta}{J_2} \cdot s^3}{(s + \omega_0)^4 + \frac{\Delta J_2}{J_2} \cdot \left(s^4 - \frac{\beta}{J_2} \cdot s^3\right)}$$

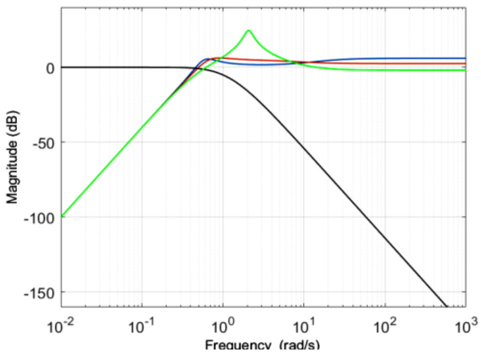
The obtained results of the study of the sensitivity function of the system synthesized by the feedback linearization method using the PI controller (Fig. 6) demonstrate that up to the cutoff frequency of the system, the LPFC of the sensitivity function is in the region of low gain coefficients, and therefore the synthesized system is also practically insensitive to such a parametric disturbance.

The transfer function of the system after substituting the values of the synthesized coefficients of the control system will have the form:

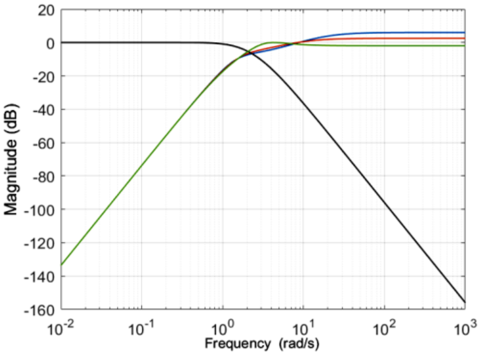
$$W^*_{fl-PI}(s) = \frac{\frac{1}{1+\delta} \cdot (2 \cdot \omega_0^3 \cdot s + \omega_0^4)}{s^4 + \left(4 \cdot \frac{1}{1+\delta} \cdot \omega_0 - \frac{\beta}{J_2} \cdot \frac{\delta}{1+\delta}\right) \cdot s^3 + 6 \cdot \omega_0^2 \cdot \frac{1}{1+\delta} \cdot s^2 + 4 \cdot \omega_0^3 \cdot \frac{1}{1+\delta} \cdot s + \omega_0^4 \cdot \frac{1}{1+\delta}}.$$



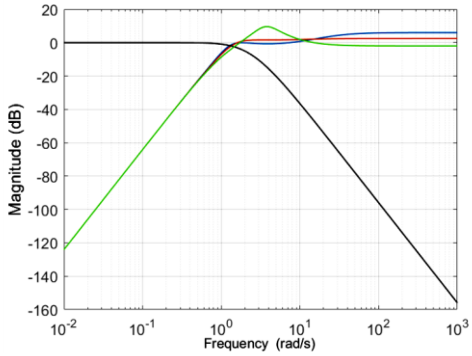
a) J1=1; J2=3; c12=10000;β=10;ω₀=1



b) J1=1; J2=1; c12=10000;β=10;ω₀=1

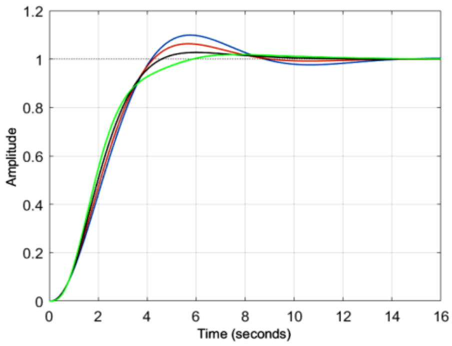


c) J1=1; J2=3; c12=10000;β=10;ω₀=2

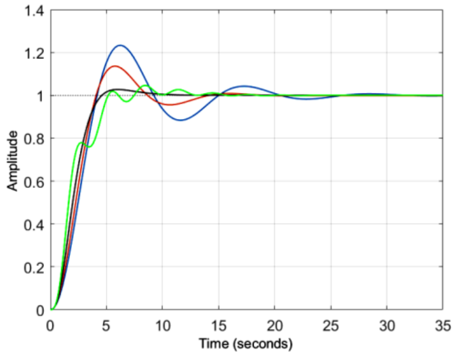


d) J1=1; J2=1; c12=10000;β=10;ω₀=2

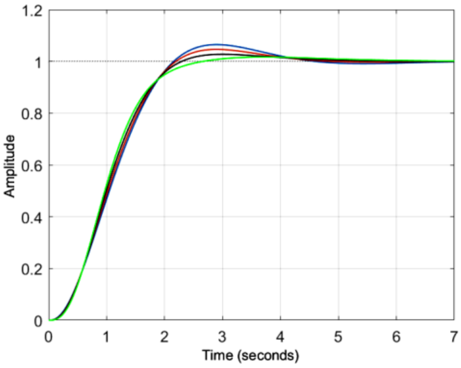
Fig. 6. LAFC synthesized by the feedback linearization method using the PI controller of the system (__) and the system sensitivity function when decreasing (__50 %, __ 25 %) and increasing (__25 %) the moment of inertia of the second mass



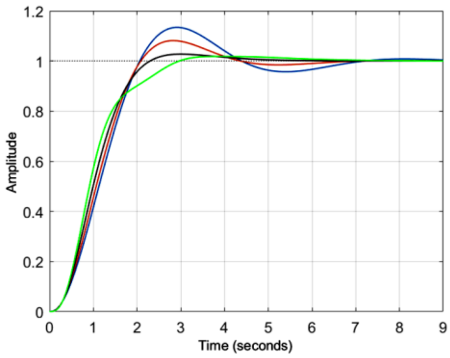
a) J1=1; J2=3; c12=10000;β=10;ω₀=1



b) J1=1; J2=1; c12=10000;β=10;ω₀=1



c) J1=1; J2=3; c12=10000;β=10;ω₀=2



d) J1=1; J2=1; c12=10000;β=10;ω₀=2

Fig. 7. Transient characteristics of the position control system synthesized by the feedback linearization method using a PI controller at a value corresponding to the parameters at the stage of system synthesis (___), as well as when decreasing (__50 %, __ 25 %) and increasing (__25 %) the moment of inertia of the second mass

The transient responses shown in Fig. 7 confirm the lower sensitivity of the system synthesized by the feedback linearization method with the use of a PI controller to an increase in the moment of inertia of the second mass, compared to the system without a PI controller and the modal control system. At the same time, a decrease in the moment of inertia relative to the value at which the feedback and PI controller coefficients were synthesized leads to an increase in overshoot and oscillations. An increase in the value of the geometric mean root contributes to reducing the system sensitivity to changes in the moment of inertia of the second mass.

Sensitivity analysis of a system synthesized by feedback linearization and application of PI$^\mu$-regulator

When using a fractional-order proportional-integral controller PI$^\mu$ in the control system and using the Caputo-Fabrizio representation to describe the fractional-order integral [26]:

$$^{CF}I^{\mu}g(t) = \frac{1}{\mu}\cdot\int_0^t e^{\frac{-(1-\mu)\cdot(t-\tau)}{\mu}}\cdot g(\tau)\,d\tau.$$

As a result of the Laplace transform we obtain: $\mathcal{L}(^{CF}I^{\mu}g(t)) = \frac{1}{\mu}\cdot\frac{1}{s+\frac{1-\mu}{\mu}}\cdot G(s) == \frac{1}{\mu\cdot s+1-\mu}\cdot G(s)$.

Then the transfer function of the controller $W_{contr}(s) = = k_p + k_i\frac{1}{\mu\cdot s+1-\mu}$. To implement integral relations for the state variables of the system, we introduce additional coordinates $y_{int}, \dot{y}_{int}, \ddot{y}_{int}$ into the system matrix. The system model will take the following form:

$$\underbrace{\frac{d}{dt}\begin{pmatrix}\omega_1\\ M_{12}\\ \omega_2\\ \varphi\\ \ddot{y}_{int}\\ \dot{y}_{int}\\ y_{int}\end{pmatrix}}_{\dot{\boldsymbol{x}}\mathbf{2}} = \underbrace{\begin{pmatrix}\boldsymbol{A_{11}} & \boldsymbol{A_{12}}\\ \boldsymbol{A_{21}} & \boldsymbol{A_{22}}\end{pmatrix}}_{A_{flPI^\mu}}\cdot\underbrace{\begin{pmatrix}\omega_1\\ M_{12}\\ \omega_2\\ \varphi\\ \ddot{y}_{int}\\ \dot{y}_{int}\\ y_{int}\end{pmatrix}}_{\boldsymbol{x}\mathbf{2}} + \underbrace{\begin{pmatrix}\frac{1}{J_1}\\ 0\\ 0\\ 0\\ 0\\ 0\\ 0\end{pmatrix}}_{B_{flPI^\mu}}\cdot M + \underbrace{\begin{pmatrix}0\\ 0\\ \frac{-1}{J_2}\\ 0\\ 0\\ 0\\ 0\end{pmatrix}}_{\boldsymbol{B1}_{flPI^\mu}}\cdot M_c$$

$$Y = \underbrace{[\mathbf{0}\quad\mathbf{0}\quad\mathbf{0}\quad\mathbf{1}\quad\mathbf{0}\quad\mathbf{0}\quad\mathbf{0}]}_{\boldsymbol{C}_{flPI^\mu}}\cdot x2$$

$$\boldsymbol{A_{11}} = \begin{bmatrix}\frac{-\beta-a_{f1}}{J_1} & \frac{-1}{J_1} & \frac{\beta}{J_1} & 0\\ c_{12} & 0 & -c_{12} & 0\\ \frac{\beta}{J_2} & \frac{1}{J_2} & \frac{-\beta-a_{f2}}{J_2} & 0\\ 0 & 0 & 1 & 0\end{bmatrix}\quad \boldsymbol{A_{12}} = \begin{bmatrix}0 & 0 & 0\\ 0 & 0 & 0\\ 0 & 0 & 0\\ 0 & 0 & 0\end{bmatrix}$$

$$\boldsymbol{A_{21}} = \begin{bmatrix}\frac{\beta}{J_2}\cdot\frac{k_i}{\mu} & \frac{1}{J_2}\cdot\frac{k_i}{\mu} & \frac{-\beta-a_{f2}}{J_2}\cdot\frac{k_i}{\mu} & 0\\ 0 & 0 & \frac{k_i}{\mu} & 0\\ 0 & 0 & 0 & \frac{k_i}{\mu}\end{bmatrix}\quad \boldsymbol{A_{22}} = \begin{bmatrix}\frac{-(1-\mu)}{\mu} & 0 & 0\\ 0 & \frac{-(1-\mu)}{\mu} & 0\\ 0 & 0 & \frac{-(1-\mu)}{\mu}\end{bmatrix}$$

Matrix transfer function of a closed-loop system synthesized by feedback linearization and using the PI$^\mu$-regulator, is defined as follows:

$$\boldsymbol{T}_{flPI^\mu}(s) = \left(s\cdot\boldsymbol{I} - \boldsymbol{A}_{flPI^\mu} + \boldsymbol{B}_{flPI^\mu}\cdot\left(\boldsymbol{C}_{flPI^\mu}\cdot\boldsymbol{A}^2_{flPI^\mu}\cdot\boldsymbol{B}_{flPI^\mu}\right)^{-1}\cdot\left(\boldsymbol{C}_{flPI^\mu}\cdot\boldsymbol{A}^3_{flPI^\mu} + k_p\cdot\boldsymbol{K}_{flPI^\mu}\times\right.\right.$$
$$\left.\left.\times\,\boldsymbol{Z}_{flPI^\mu} + \boldsymbol{K}_{flPI^\mu}\cdot\boldsymbol{Z1}_{flPI^\mu}\right)\right)\cdot\boldsymbol{B}_{flPI^\mu}\cdot\left(\boldsymbol{C}_{flPI^\mu}\cdot\boldsymbol{A}^2_{flPI^\mu}\cdot\boldsymbol{B}_{flPI^\mu}\right)^{-1},$$

where s is the Laplace operator; I_{5x5} is the identity matrix; $\boldsymbol{K}_{flPI^\mu} = [k_{1flrm}\quad k_{2flrm}\quad k_{3flrm}]$ is the vector of feedback coefficients for the variables $y, \dot{y}, \ddot{y},$; $\boldsymbol{Z}_{flPI^\mu} = \begin{bmatrix}\boldsymbol{C}_{flPI^\mu}\\ \boldsymbol{C}_{flPI^\mu}\cdot\boldsymbol{A}_{flPI^\mu}\\ \boldsymbol{C}_{flPI^\mu}\cdot\boldsymbol{A}^2_{flPI^\mu}\end{bmatrix}$ is the matrix of the connection between the variables $y, \dot{y}, \ddot{y},$ and the state vector of the system *x*;

$\boldsymbol{Z}_{\boldsymbol{flPI^{\mu}}} = \begin{bmatrix} \boldsymbol{C}_{\boldsymbol{flPI^{\mu}}} \\ \boldsymbol{C}_{\boldsymbol{flPI^{\mu}}} \cdot \boldsymbol{A}_{\boldsymbol{flPI^{\mu}}} \\ \boldsymbol{C}_{\boldsymbol{flPI^{\mu}}} \cdot \boldsymbol{A}^{\boldsymbol{2}}_{\boldsymbol{flPI^{\mu}}} \end{bmatrix}$ is the matrix of the connection between the variables $y, \dot{y}, \ddot{y}$ of the state vector of the system $x2$; and $\boldsymbol{C1}_{\boldsymbol{flPI^{\mu}}} = [\boldsymbol{0} \quad \boldsymbol{0} \quad \boldsymbol{0} \quad \boldsymbol{0} \quad \boldsymbol{0} \quad \boldsymbol{0} \quad \boldsymbol{1}]$, $\boldsymbol{C2}_{\boldsymbol{flPI}} = [\boldsymbol{0} \quad \boldsymbol{0} \quad \boldsymbol{0} \quad \boldsymbol{0} \quad \boldsymbol{0} \quad \boldsymbol{1} \quad \boldsymbol{0}]$, $\boldsymbol{C3}_{\boldsymbol{flPI}} = \; = [\boldsymbol{0} \quad \boldsymbol{0} \quad \boldsymbol{0} \quad \boldsymbol{0} \quad \boldsymbol{1} \quad \boldsymbol{0} \quad \boldsymbol{0}]$.

The system transfer function will be as follows:

$$W_{fl-PI^{\mu}}(s) = \frac{k_p \cdot s + (k_p \cdot (1-\mu) + k_i) \cdot \frac{1}{\mu}}{s^4 + \left(k_p \cdot k_{3flr} + \frac{1-\mu}{\mu}\right) s^3 + \left(k_p \cdot k_{2flr} + (k_p \cdot (1-\mu) + k_i) \cdot \frac{k_{3flr}}{\mu}\right) s^2 + \to \\ \to + \left(k_p \cdot k_{1flr} + (k_p \cdot (1-\mu) + k_i) \cdot \frac{k_{2flr}}{\mu}\right) s + (k_p \cdot (1-\mu) + k_i) \cdot \frac{k_{1flr}}{\mu}}.$$

In the case of setting to the standard binomial form, the system of equations for finding the controller coefficients and feedbacks on the state variables will have the following form:

$$\begin{cases} k_p \cdot k_{3flr} + \dfrac{1-\mu}{\mu} = 4 \cdot \omega_0 \\ k_p \cdot k_{2flr} + (k_p \cdot (1-\mu) + k_i) \cdot \dfrac{k_{3flr}}{\mu} = 6 \cdot {\omega_0}^2 \\ k_p \cdot k_{1flr} + (k_p \cdot (1-\mu) + k_i) \cdot \dfrac{k_{2flr}}{\mu} = 4 \cdot {\omega_0}^3 \\ (k_p \cdot (1-\mu) + k_i) \cdot \dfrac{k_{1flr}}{\mu} = {\omega_0}^4 \end{cases}$$

This system of equations allows for finding the coefficients of the controller and feedback loops as functions $\omega_0$and μ. When changing the moment of inertia of the second mass, the system matrix will have the form

$$\boldsymbol{A}^*_{\boldsymbol{flPI^{\mu}}} = \begin{pmatrix} \boldsymbol{A}^*_{\boldsymbol{11}} & \boldsymbol{A}_{\boldsymbol{12}} \\ \boldsymbol{A}^*_{\boldsymbol{21}} & \boldsymbol{A}_{\boldsymbol{22}} \end{pmatrix} \qquad \boldsymbol{A}^*_{\boldsymbol{11}} = \begin{bmatrix} \dfrac{-\beta - a_{f1}}{J_1} & \dfrac{-1}{J_1} & \dfrac{\beta}{J_1} & 0 \\ c_{12} & 0 & -c_{12} & 0 \\ \dfrac{\beta}{J_2 + \Delta J_2} & \dfrac{1}{J_2 + \Delta J_2} & \dfrac{-\beta - a_{f2}}{J_2 + \Delta J_2} & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

$$\boldsymbol{A}^*_{\boldsymbol{21}} = \begin{bmatrix} \dfrac{\beta}{J_2 + \Delta J_2} \cdot \dfrac{k_i}{\mu} & \dfrac{1}{J_2 + \Delta J_2} \cdot \dfrac{k_i}{\mu} & \dfrac{-\beta - a_{f2}}{J_2 + \Delta J_2} \cdot \dfrac{k_i}{\mu} & 0 \\ 0 & 0 & \dfrac{k_i}{\mu} & 0 \\ 0 & 0 & 0 & \dfrac{k_i}{\mu} \end{bmatrix}$$

Accordingly, the matrix transfer function is defined as follows:

$$\boldsymbol{T}_{\boldsymbol{flPI^{\mu}}}(s) = (s \cdot \boldsymbol{I} - \boldsymbol{A}^*_{\boldsymbol{flPI^{\mu}}} + \boldsymbol{B}_{\boldsymbol{flPI^{\mu}}} \cdot \left(\boldsymbol{C}_{\boldsymbol{flPI^{\mu}}} \cdot \boldsymbol{A}^{\boldsymbol{2}}_{\boldsymbol{flPI^{\mu}}} \cdot \boldsymbol{B}_{\boldsymbol{flPI^{\mu}}}\right)^{-1} \cdot \; (\boldsymbol{C}_{\boldsymbol{flPI^{\mu}}} \cdot \boldsymbol{A}^{\boldsymbol{3}}_{\boldsymbol{flPI^{\mu}}} + k_p \cdot \boldsymbol{K}_{\boldsymbol{flPI^{\mu}}} \times \\ \times \boldsymbol{Z}^*_{\boldsymbol{flPI^{\mu}}} + \boldsymbol{K}_{\boldsymbol{flPI^{\mu}}} \cdot \boldsymbol{Z1}_{\boldsymbol{flPI^{\mu}}})) \cdot \boldsymbol{B}_{\boldsymbol{flPI^{\mu}}} \cdot \left(\boldsymbol{C}_{\boldsymbol{flPI^{\mu}}} \cdot \boldsymbol{A}^{\boldsymbol{2}}_{\boldsymbol{flPI^{\mu}}} \cdot \boldsymbol{B}_{\boldsymbol{flPI^{\mu}}}\right)^{-1},$$

where $\boldsymbol{Z}^*_{\boldsymbol{flPI^{\mu}}} = \begin{bmatrix} \boldsymbol{C}_{\boldsymbol{flPI^{\mu}}} \\ \boldsymbol{C}_{\boldsymbol{flPI^{\mu}}} \cdot \boldsymbol{A}^*_{\boldsymbol{flPI^{\mu}}} \\ \boldsymbol{C}_{\boldsymbol{flPI^{\mu}}} \cdot \boldsymbol{A}^{*\boldsymbol{2}}_{\boldsymbol{flPI^{\mu}}} \end{bmatrix}$ is the matrix of the connection between the variables $y, \dot{y}, \ddot{y}$ and the state vector of the system $x2$.

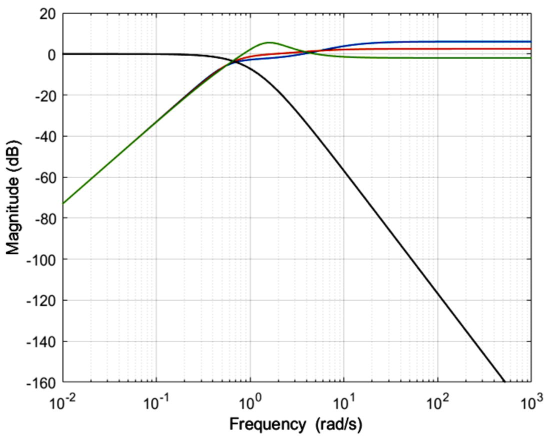
Researched transfer function is as follows:

$$W^*{}_{fl-PI^{\mu}}(s) = \frac{k_p \cdot s + \left(k_p \cdot (1-\mu) + k_i\right) \cdot \frac{1}{\mu}}{\begin{array}{c} s^4 + \left(k_p \cdot k_{3flr} + \frac{1-\mu}{\mu}\right) s^3 + \left(k_p \cdot k_{2flr} + \left(k_p \cdot (1-\mu) + k_i\right) \cdot \frac{k_{3flr}}{\mu}\right) s^2 + \rightarrow \\ \rightarrow + \left(k_p \cdot k_{1flr} + \left(k_p \cdot (1-\mu) + k_i\right) \cdot \frac{k_{2flr}}{\mu}\right) s + \left(k_p \cdot (1-\mu) + k_i\right) \cdot \frac{k_{1flr}}{\mu} + \rightarrow \\ \rightarrow \frac{\Delta J_2}{J_2} \cdot \left(s^4 + \left(\frac{1-\mu}{\mu} - \frac{b}{J_2}\right) s^3 + \left(-\frac{1-\mu}{\mu} \cdot \frac{b}{J_2}\right) s^2\right) \end{array}}$$

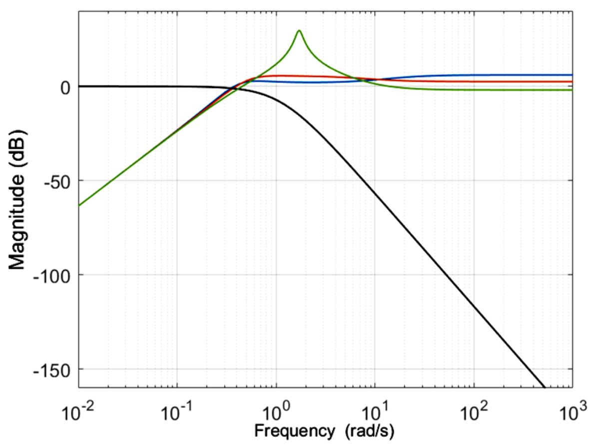
and the sensitivity function, after substituting the found values of the feedback coefficients and the controller coefficients, will have the form:

$$Sens_{fl-PI^{\mu}}(s, J_2) = \frac{s^4 + \left(\frac{1-\mu}{\mu} - \frac{b}{J_2}\right) s^3 + \left(-\frac{1-\mu}{\mu} \cdot \frac{b}{J_2}\right) s^2}{(s+\omega_0)^4 + \frac{\Delta J_2}{J_2} \cdot \left(s^4 + \left(\frac{1-\mu}{\mu} - \frac{b}{J_2}\right) s^3 + \left(-\frac{1-\mu}{\mu} \cdot \frac{b}{J_2}\right) s^2\right)}.$$

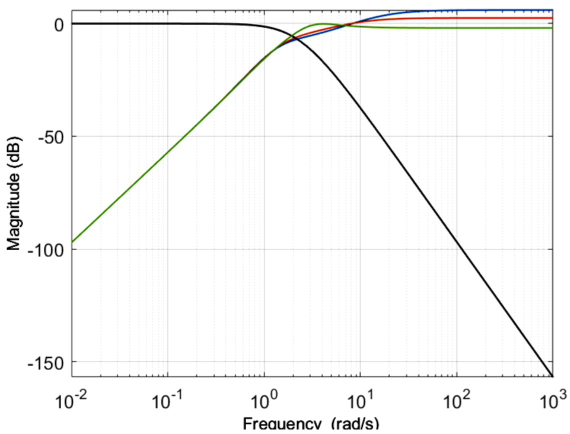
The dependencies shown in Fig. 8, 9 allow for stating that the use of a proportional-integral fractional-order controller PI^{μ} for the parameters $\in [0.5; 1[$ under study provides a system whose sensitivity lies in the range from the sensitivity of the system synthesized by the feedback linearization method to the sensitivity of the system synthesized by the feedback linearization method and the use of a PI controller.



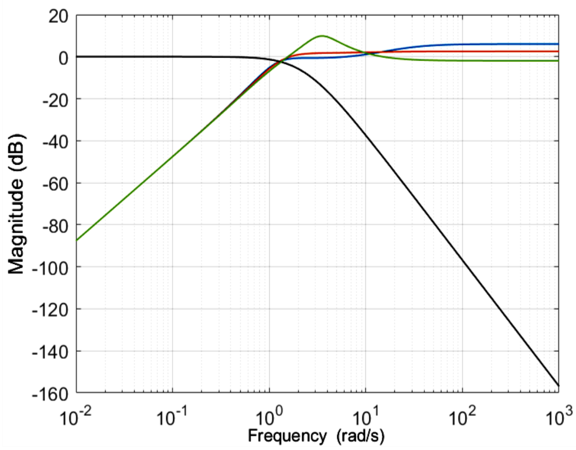
a) J1=1; J2=3; c12=10000;β=10;ω0=1



b) J1=1; J2=1; c12=10000;β=10;ω0=1



c) J1=1; J2=3; c12=10000;β=10;ω0=2



d) J1=1; J2=1; c12=10000;β=10;ω0=2

Fig. 8. LPFC synthesized by the feedback linearization method using the PI0.6 system controller (__) and the system sensitivity function when decreasing (__50 %, __ 25 %) and increasing (__25 %) the moment of inertia of the second mass

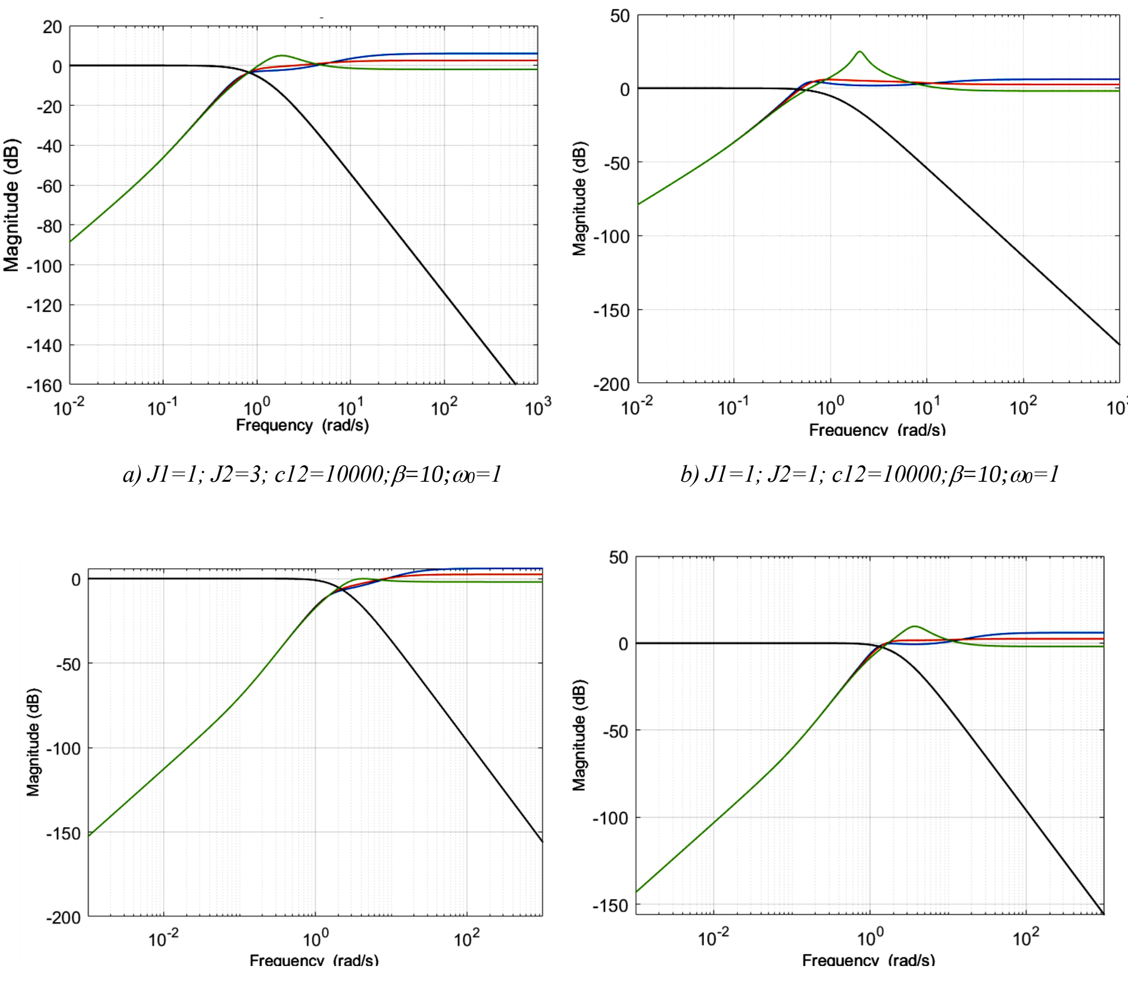
The transfer function of the system after substituting the calculated coefficients will be as follows

$$W^{*}_{fl-PI}(s)=\frac{\frac{1}{1+\delta}\cdot\left(\left(\frac{4\cdot\omega_0^{\,3}}{3}-\frac{2\cdot\omega_0^{\,6}}{9\cdot R}+R\right)\cdot s+\omega_0^4\right)}{s^4+\left(4\cdot\frac{1}{1+\delta}\cdot\omega_0+\left(\frac{1-\mu}{\mu}-\frac{b}{J_2}\right)\cdot\frac{\delta}{1+\delta}\right)\cdot s^3+\left(6\cdot\omega_0^2\cdot\frac{1}{1+\delta}-\frac{1-\mu}{\mu}\cdot\frac{b}{J_2}\cdot\frac{\delta}{1+\delta}\right)\cdot s^2\rightarrow}$$

$$\rightarrow+4\cdot\omega_0^3\cdot\frac{1}{1+\delta}\cdot s+\omega_0^4\cdot\frac{1}{1+\delta}$$

where

$$R=\sqrt[3]{\frac{10\cdot\omega_0^{\,9}}{27}-\frac{\frac{1-\mu}{\mu}\cdot\omega_0^{\,8}}{2}+\frac{\omega_0^{\,8}}{2}\cdot\sqrt{\frac{2\cdot\left(\frac{1-\mu}{\mu}\right)^2+\left(5\cdot\frac{1-\mu}{\mu}-4\cdot\omega_0\right)^2}{27}}}$$



a) J1=1; J2=3; c12=10000;β=10;ω0=1

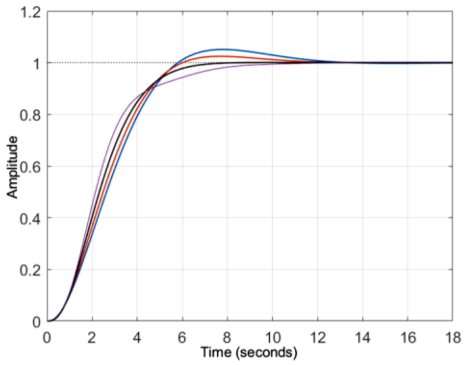
b) J1=1; J2=1; c12=10000;β=10;ω0=1

c) J1=1; J2=3; c12=10000;β=10;ω0=2

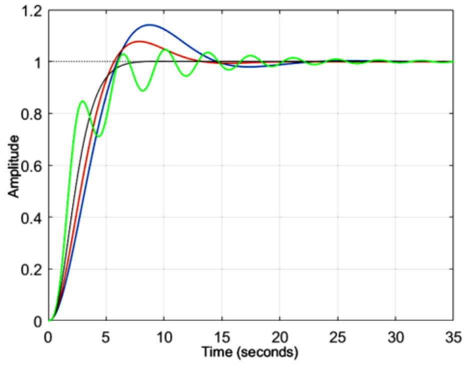
d) J1=1; J2=1; c12=10000;β=10;ω0=2

Fig. 9. LAFC synthesized by the feedback linearization method using the PI0.9-regulator of the system (__) and the sensitivity function of the system when decreasing (__50 %, __ 25 %) and increasing (__25 %) the moment of inertia of the second mass

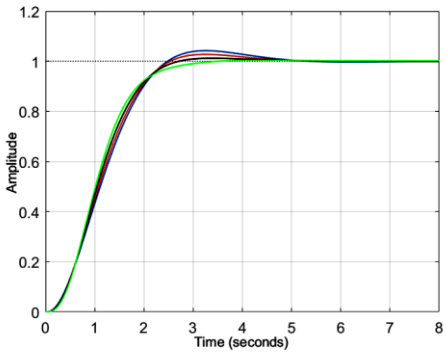
The transient characteristics of the systems with fractional-order proportional-integral controllers PIμ shown in Fig. 10, 11 demonstrate the possibility of obtaining a system with slightly smaller overshoots when the moment of inertia of the second mass is reduced compared to a system with a traditional PI controller. This is primarily explained by the smaller magnitude of overshoot when the systems are precisely tuned.



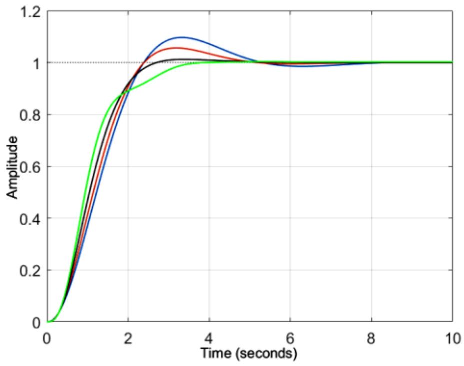
a) J1=1; J2=3; c12=10000;β=10;ω0=1



b J1=1; J2=1; c12=10000;β=10;ω0=1

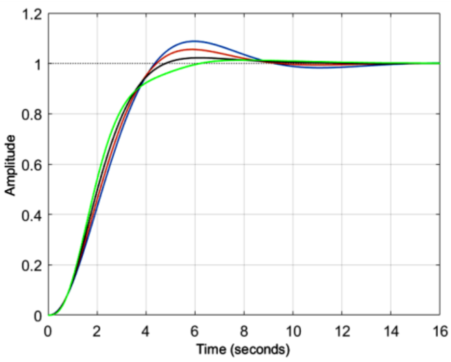


c) J1=1; J2=3; c12=10000;β=10;ω0=2

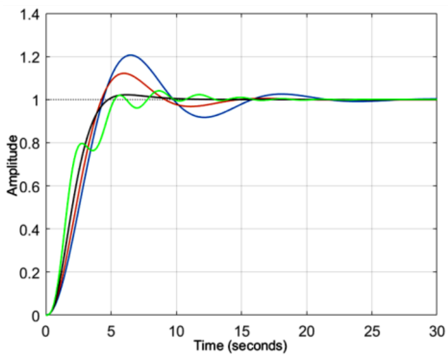


d) J1=1; J2=1; c12=10000;β=10;ω0=2

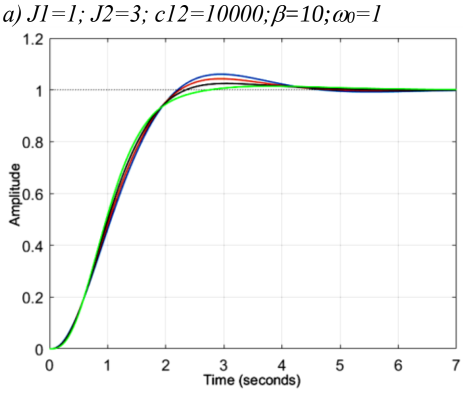
Fig. 10. Transient characteristics of the positional control system synthesized by the feedback linearization method using a PI0.6 controller at a value corresponding to the parameters at the stage of system synthesis (___), as well as when decreasing (__50 %, __ 25 %) and increasing (__25 %) the moment of inertia of the second mass



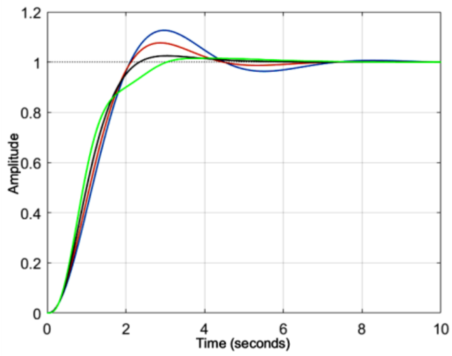
a) J1=1; J2=3; c12=10000;β=10;ω0=1



b) J1=1; J2=1; c12=10000;β=10;ω0=1



c) J1=1; J2=3; c12=10000;β=10;ω0=2



d) J1=1; J2=1; c12=10000;β=10;ω0=2

Fig. 11. Transient characteristics of the positional control system synthesized by the feedback linearization method using a PI0.9 controller at a value corresponding to the parameters at the stage of system synthesis (___), as well as when decreasing (__50 %, __ 25 %) and increasing (__25 %) moment of inertia of the second mass. Mu = 0.65 kp = 1.606

Conclusions

The results of the performed studies demonstrate high sensitivity to changes in the moment of inertia of the second mass of systems synthesized by the modal control method. Considering that such a change in the moment of inertia of the second mass can be caused both by the features of the technological mode and by an error in identifying the parameters of the two-mass model of the mechanical system, it is more expedient to use the feedback linearization method for the synthesis of the control effect.

Synthesis of the control system by the feedback linearization method using a PI controller reduces the sensitivity of the system to changes in the moment of inertia of the second mass even in comparison with the system synthesized by the feedback linearization method, however, while providing better dynamic characteristics, it causes larger overshoots of the initial system coordinate.

Application of the proportional-integral fractional-order controller PIμ allows improving the system characteristics in terms of reducing the sensitivity to changes in the moment of inertia of the second mass and the magnitude of the overshoot of the initial coordinate, with a slight deterioration in the dynamic characteristics $\mu \in [0.6; 0.7]$ compared to the system synthesized by the method of feedback linearization using a PI controller.

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АНАЛІЗ ЧУТЛИВОСТІ СИСТЕМ КЕРУВАННЯ, СИНТЕЗОВАНИХ МЕТОДАМИ FEEDBACK CONTROL, ДО ЗМІНИ МОМЕНТУ ІНЕРЦІЇ ДРУГОЇ МАСИ ДВОМАСОВОЇ ПОЗИЦІЙНОЇ СИСТЕМИ

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В роботі запропоновано комплексний підхід до аналізу чутливості систем керування за змінними стану. Продемонстровано переваги систем, синтезованих методом feedback linearization, перед системою, синтезованою методом модального керування як з точки зору чутливості до зміни моменту інерції другої маси, так і якості регулювання. Проаналізовано вплив застосування ПІ-регулятора та ПІ$^{\mu}$-регулятора в системах, синтезованих методом feedback linearization, як на чутливість системи до зміни моменту інерції другої маси, так і швидкодію та перерегулювання вихідної координати.

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