

# Degree-based energies of commuting graph for dihedral groups

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Commuting graph for a finite group  $G$ , denoted by  $\Gamma_G$ , with its set of vertices  $G \setminus Z(G)$ , where  $Z(G)$  is the center of  $G$ , is a graph with  $v_p, v_q \in G \setminus Z(G)$ ,  $v_p \neq v_q$ , are adjacent whenever  $v_p v_q = v_q v_p$ . In recent years, there has been significant research into the energy of graphs, particularly focusing on matrices associated with the degree of vertices. Therefore, motivated by that, our study elaborates on the energy of  $\Gamma_G$  for dihedral groups of order  $2n$ ,  $D_{2n}$ , concerning some graph matrices related to the degree of elements of  $D_{2n} \setminus Z(D_{2n})$  and examine the correlation between those energies. The matrices involved are known as geometric-arithmetic, symmetric division deg, degree exponent, inverse sum indeg and Sombor matrices. Based on these five matrices, it is found that the lowest graph energy is the geometric-arithmetic energy of  $\Gamma_G$  whilst the highest is the degree exponent energy. Furthermore, the geometric-arithmetic, symmetric division deg, and degree exponent energies are always positive even integers. In contrast, the inverse sum indeg energy is a positive integer that can be either even or odd. Meanwhile, the Sombor energy is never an odd integer.

**Keywords:** commuting graph; energy of graph; degree-based matrices; dihedral group.

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## 1. Introduction

The non-abelian dihedral group of order  $2n$  is a group with  $2n$  elements, where  $n$  is 3 or greater. The operation is composition and it is referred to as  $D_{2n} = \langle a, b : a^n = b^2 = e, bab = a^{-1} \rangle$  [1]. The center of  $D_{2n}$  is called  $Z(D_{2n})$ , in the form of  $Z(D_{2n}) = \{e\}$  for odd  $n$ , or for even  $n$ , it is  $\{e, a^{\frac{n}{2}}\}$ . Moreover, in  $D_{2n}$ , the centralizer of  $a^i$  is represented by  $C_{D_{2n}}(a^i) = \{a^j : 1 \leq j \leq n\}$ . Meanwhile, for the other elements of  $D_{2n}$ , we have two cases which are  $C_{D_{2n}}(a^i b) = \{e, a^i b\}$  for odd  $n$ , or  $C_{D_{2n}}(a^i b) = \{e, a^{\frac{n}{2}}, a^i b, a^{\frac{n}{2}+i} b\}$  for even  $n$ .

Commuting graph  $\Gamma_G$  for a finite group  $G$ , possesses a set of vertices  $G \setminus Z(G)$ , and  $v_p, v_q \in G \setminus Z(G)$  are adjacent whenever  $v_p v_q = v_q v_p$ , where  $Z(G)$  is the center of  $G$  [3]. Several discussions of this graph with a symmetric group of degree  $n$  as the set of vertices have been conducted, see [7]. The energy of  $\Gamma_G$  equals the total of the absolute values of all of its eigenvalues. Gutman [5] was the first to propose this definition in 1978. According to [2] and [8], the graph energy is never an odd integer.

In recent years, research on graph energy development has marked an increase in graph matrices studies. Despite the adjacency matrices, much work has been done on new graph matrices. Determination of the energy bounds for a simple graph associated with several degree-based matrices was obtained, for instance, geometric-arithmetic energy [11], symmetric division deg energy [9], inverse sum indeg [16], and Sombor energy [6]. Besides, Ramane and Shinde [10] introduced the degree exponent matrix and stated its characteristic polynomial for regular graphs.

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The Seidel Laplacian and Seidel Signless Laplacian, degree exponent sum, maximum and minimum degree energies of  $\Gamma_G$  for  $D_{2n}$  were shown in [12, 14, 15]. Meanwhile, Romdhini and Nawawi formulated the energy of non-commuting graphs for  $D_{2n}$  by analyzing the eigenvalues of the Sombor matrix [13]. Consequently, the research aims of this study were to delineate a comprehensive formula of commuting graph energy for dihedral groups based on the eigenvalues of a geometric-arithmetic matrix, symmetric division deg matrix, degree exponent matrix, inverse sum indeg matrix, and Sombor matrix.

This paper is divided into 5 parts. In Section 2, several theorems and lemmas are introduced to construct matrices and solve their characteristic polynomials, as well as to compare degree-based energies. Section 3 is primarily concerned with the energy formula of the commuting graph for  $D_{2n}$  resulting from seven different types of graph matrices. The discussion of the relationship between those energies is given in Section 4. In the end, in Section 5, we provide brief concluding remarks.

## 2. Preliminaries

In this part we start with five degree-based matrix definitions, considering  $d_{v_p}$  as the degree of  $v_p$ .

**Definition 1 (Ref. [11]).** The  $n \times n$  geometric-arithmetic matrix of  $\Gamma_G$ ,  $\text{GA}(\Gamma_G) = [ga_{pq}]$  in which  $(p, q)$ -th entry is

$$ga_{pq} = \begin{cases} \frac{2\sqrt{d_{v_p}d_{v_q}}}{d_{v_p} + d_{v_q}}, & \text{if } v_p \neq v_q \text{ are adjacent,} \\ 0, & \text{otherwise.} \end{cases}$$

**Definition 2 (Ref. [9]).** The  $n \times n$  symmetric division deg matrix of  $\Gamma_G$ ,  $\text{SDD}(\Gamma_G) = [sdd_{pq}]$  in which  $(p, q)$ -th entry is

$$sdd_{pq} = \begin{cases} \frac{\min\{d_{v_p}, d_{v_q}\}}{\max\{d_{v_p}, d_{v_q}\}} + \frac{\max\{d_{v_p}, d_{v_q}\}}{\min\{d_{v_p}, d_{v_q}\}}, & \text{if } v_p \neq v_q \text{ are adjacent,} \\ 0, & \text{otherwise.} \end{cases}$$

**Definition 3 (Ref. [10]).** The  $n \times n$  degree exponent matrix of  $\Gamma_G$ ,  $\text{DE}(\Gamma_G) = [de_{pq}]$  in which  $(p, q)$ -th entry is

$$de_{pq} = \begin{cases} d_{v_p}^{d_{v_q}}, & \text{if } p \neq q, \\ 0, & \text{if } p = q. \end{cases}$$

**Definition 4 (Ref. [16]).** The  $n \times n$  inverse sum indeg matrix of  $\Gamma_G$ ,  $\text{ISI}(\Gamma_G) = [isi_{pq}]$  in which  $(p, q)$ -th entry is

$$isi_{pq} = \begin{cases} \frac{d_{v_p}d_{v_q}}{d_{v_p} + d_{v_q}}, & \text{if } v_p \neq v_q \text{ are adjacent,} \\ 0, & \text{otherwise.} \end{cases}$$

**Definition 5 (Ref. [6]).** The  $n \times n$  Sombor matrix of  $\Gamma_G$ ,  $\text{S}(\Gamma_G) = [s_{pq}]$  in which  $(p, q)$ -th entry

$$s_{pq} = \begin{cases} \sqrt{d_{v_p}^2 + d_{v_q}^2}, & \text{if } v_p \neq v_q \text{ are adjacent,} \\ 0, & \text{otherwise.} \end{cases}$$

Moreover, the spectrum of  $\Gamma_G$  is the list of eigenvalues  $\lambda_1, \lambda_2, \dots, \lambda_n$ , with their respective multiplicities  $k_1, k_2, \dots, k_n$ , denoted by  $\text{Spec}(\Gamma_G) = \{\lambda_1^{k_1}, \lambda_2^{k_2}, \dots, \lambda_n^{k_n}\}$ . Now we present some previous results to prove the theorems obtained in Section 3. In determining the energy of the graph, it is necessary to formulate the characteristic polynomial of  $\Gamma_G$ . Two essential results help to compute it as follows.

**Lemma 1 (Ref. [10]).** Suppose real numbers  $a, b, c$ , and  $d$ . Given  $J_n$  is a matrix of size  $n \times n$  in which all components are 1. Then the determinant of

$$\begin{vmatrix} (\lambda + a)I_{n_1} - aJ_{n_1} & -cJ_{n_1 \times n_2} \\ -dJ_{n_2 \times n_1} & (\lambda + b)I_{n_2} - bJ_{n_2} \end{vmatrix}$$

can be simplified as

$$(\lambda + a)^{n_1-1}(\lambda + b)^{n_2-1}((\lambda - (n_1 - 1)a)(\lambda - (n_2 - 1)b) - n_1 n_2 cd),$$

where  $1 \leq n_1, n_2 \leq n$  and  $n_1 + n_2 = n$ .

**Theorem 1 (Ref. [4]).** Given a square matrix  $M = \begin{bmatrix} A & B \\ C & D \end{bmatrix}$  that can be divided into 4 blocks, where  $A$  is a square matrix with  $|A| \neq 0$ , then

$$|M| = \begin{vmatrix} A & B \\ O & D - CA^{-1}B \end{vmatrix} = |A||D - CA^{-1}B|.$$

This paper concentrates on dihedral groups,  $D_{2n}$ . Let  $G_1 = \{a^i : 1 \leq i \leq n\} \setminus Z(D_{2n})$  and  $G_2 = \{a^i b : 1 \leq i \leq n\}$ . Results regarding the degree of each vertex in  $\Gamma_G$  for  $G = G_1 \cup G_2$  is presented as follows.

**Theorem 2 (Ref. [15]).** Let  $G = G_1 \cup G_2$ , then in  $\Gamma_G$ ,

1.  $d_{a^i} = \begin{cases} n-2, & \text{if } n \text{ is odd,} \\ n-3, & \text{if } n \text{ is even,} \end{cases}$  and
2.  $d_{a^i b} = \begin{cases} 0, & \text{if } n \text{ is odd,} \\ 1, & \text{if } n \text{ is even.} \end{cases}$

As a result, the subsequent theorem demonstrates the isomorphism between the commuting graph and the standard type of graphs.

**Theorem 3 (Ref. [15]).** In  $\Gamma_G$ , we have

1.  $\Gamma_G \cong K_m$ , where  $G = G_1$  and  $m = |G_1|$ , and
2. for  $G = G_2$ ,  $\Gamma_G \cong \begin{cases} \bar{K}_n, & \text{for odd } n, \\ 1\text{-regular graph,} & \text{for even } n. \end{cases}$

Furthermore, we use the following two theorems to compare the results obtained in this paper.

**Theorem 4 (Ref. [15]).** Let  $G = G_1 \cup G_2$ . The degree exponent sum energy for  $\Gamma_G$  is

1. for  $n$  is odd,  $E_{\text{DES}}(\Gamma_G) = 4(n-2)^{n-1} + 4(n-1)$ , and
2. for  $n$  is even,  $E_{\text{DES}}(\Gamma_G) = \begin{cases} 20, & \text{if } n = 4, \\ 4(n-3)^{n-2} + 4(n-1), & \text{if } n > 4. \end{cases}$

**Theorem 5 (Ref. [14]).** Let  $G = G_1 \cup G_2$ . The maximum degree and minimum degree energies for  $\Gamma_G$  are

1. for  $n$  is odd,  $E_{\text{MaxD}}(\Gamma_G) = E_{\text{MinD}}(\Gamma_G) = 2(n-2)^2$ , and
2. for  $n$  is even,  $E_{\text{MaxD}}(\Gamma_G) = E_{\text{MinD}}(\Gamma_G) = 2(n-3)^2 + n$ .

### 3. Results

This section will present the results of commuting graph energy for  $D_{2n}$  using the corresponding graph matrices related to the degree of vertices of  $\Gamma_G$ . This only applies to  $n \geq 3$ , as  $D_{2n}$  is abelian for  $n = 1$  and  $n = 2$ .

#### 3.1. Geometric-arithmetic energy

**Theorem 6.** Let  $G = G_1 \cup G_2$ . The geometric-arithmetic energy of  $\Gamma_G$  is

1. for  $n$  is odd,  $E_{GA}(\Gamma_G) = 2(n-2)$ , and
2. for  $n$  is even,  $E_{GA}(\Gamma_G) = 2(n-3) + n$ .

**Proof.**

1. As is well acknowledged for the odd  $n$ ,  $Z(D_{2n}) = \{e\}$  indicates that there are  $2n-1$  vertices in  $\Gamma_G$ . We write the set  $G_1$  as  $\{a, a^2, \dots, a^{n-1}\}$  and  $G_2$  as  $\{b, ab, a^2b, \dots, a^{n-1}b\}$ . From the fact that  $C_{D_{2n}}(a^i) = \{e, a, a^2, \dots, a^{n-1}\}$ , thus the vertex  $a^i$ , for  $i = 1, 2, \dots, n-1$  always has an edge

with all members of  $G_1$ , nonetheless, it is not adjacent to every vertex in  $G_2$ . The centralizer of  $a^i b$  is  $\{a, a^i b\}$  is indicating that for  $i = 1, 2, \dots, n$ , the vertex  $a^i b$  is not adjacent with all other components of  $G_1 \cup G_2$ . Considering Theorem 2 (1), we have  $d_{a^i} = n - 2$  and  $d_{a^i b} = 0$ , for all  $1 \leq i \leq n$ . Then for the adjacent two distinct vertices, the entries of  $\text{GA}(\Gamma_G)$  are  $\frac{2\sqrt{(n-2)(n-2)}}{n-2+n-2} = 1$ , and zero otherwise. Now from Definition 1 and Theorem 3 (1), we acquire the geometric-arithmetic matrix of  $\Gamma_G$  is a  $(2n - 1) \times (2n - 1)$  matrix

$$\text{GA}(\Gamma_G) = \begin{bmatrix} (J - I)_{n-1} & 0_{(n-1) \times n} \\ 0_{n \times (n-1)} & 0_n \end{bmatrix}. \quad (1)$$

The characteristic formula of  $\text{GA}(\Gamma_G)$  is

$$P_{\text{GA}(\Gamma_G)}(\lambda) = \begin{vmatrix} (\lambda + 1)I_{n-1} - J_{n-1} & 0_{(n-1) \times n} \\ 0_{n \times (n-1)} & \lambda I_n \end{vmatrix}.$$

Employing Lemma 1, with  $a = 1$ ,  $b = c = d = 0$ ,  $n_1 = n - 1$  and  $n_2 = n$ , we then get

$$P_{\text{GA}(\Gamma_G)}(\lambda) = (\lambda + 1)^{n-2} \lambda^n (\lambda - (n - 2)).$$

It implies that GA-spectrum of  $\Gamma_G$  is

$$\text{Spec}(\Gamma_G) = \{(n - 2)^1, (0)^n, (-1)^{n-2}\}.$$

By direct calculation, we obtain the following geometric-arithmetic energy of  $\Gamma_G$ :

$$E_{\text{GA}}(\Gamma_G) = (n - 2)|-1| + (n)|0| + |n - 2| = 2(n - 2). \quad (2)$$

- Given that  $n$  is even where  $Z(D_{2n}) = \{e, a^{\frac{n}{2}}\}$  means that we have  $2n - 2$  vertices in  $\Gamma_G$ , for  $G = G_1 \cup G_2$ , with  $n - 2$  vertices  $a^i$ ,  $1 \leq i < \frac{n}{2}$ ,  $\frac{n}{2} < i < n$ , and  $n$  vertices  $a^i b$ ,  $1 \leq i \leq n$ . We designate the set  $G_1$  as  $\{a, a^2, \dots, a^{\frac{n}{2}-1}, a^{\frac{n}{2}+1}, \dots, a^{n-1}\}$  and  $G_2$  as  $\{b, ab, a^2b, \dots, a^{n-1}b\}$ . Repeated application of Theorem 2 (2), we have  $d_{a^i} = n - 3$  and  $d_{a^i b} = 1$ . On the other hand, to construct  $\text{GA}(\Gamma_G)$ , we use Definition 1 and Theorem 3 (2), so we obtain  $(2n - 2) \times (2n - 2)$  matrix as follows:

$$\text{GA}(\Gamma_G) = \begin{bmatrix} (J - I)_{n-2} & 0_{(n-2) \times \frac{n}{2}} & 0_{(n-2) \times \frac{n}{2}} \\ 0_{\frac{n}{2} \times (n-2)} & 0_{\frac{n}{2}} & I_{\frac{n}{2}} \\ 0_{\frac{n}{2} \times (n-2)} & I_{\frac{n}{2}} & 0_{\frac{n}{2}} \end{bmatrix}. \quad (3)$$

Therefore, the characteristic formula of  $\text{GA}(\Gamma_G)$  is derived from the subsequent determinant:

$$P_{\text{GA}(\Gamma_G)}(\lambda) = \begin{vmatrix} (\lambda + 1)I_{n-2} - J_{n-2} & 0_{(n-2) \times \frac{n}{2}} & 0_{(n-2) \times \frac{n}{2}} \\ 0_{\frac{n}{2} \times (n-2)} & \lambda I_{\frac{n}{2}} & -I_{\frac{n}{2}} \\ 0_{\frac{n}{2} \times (n-2)} & -I_{\frac{n}{2}} & \lambda I_{\frac{n}{2}} \end{vmatrix}. \quad (4)$$

According to Theorem 1 with

$$A = [(\lambda + 1)I_{n-2} - J_{n-2}], \quad (5)$$

$B = [0_{(n-2) \times n}]$ ,  $C = [0_{n \times (n-2)}]$ , and

$$D = \begin{bmatrix} \lambda I_{\frac{n}{2}} & -I_{\frac{n}{2}} \\ -I_{\frac{n}{2}} & \lambda I_{\frac{n}{2}} \end{bmatrix}, \quad (6)$$

then Equation (4) is the form  $P_{\text{GA}(\Gamma_G)}(\lambda) = |A||D|$  (since  $C = 0$ ). Now we consider the matrix determinant in Equation (5). By using Lemma 1 with  $a = b = c = d = 1$ , and  $n_1 = n_2 = \frac{n-2}{2}$ , it is determined that

$$|A| = (\lambda + 1)^{n-3} (\lambda - (n - 3)). \quad (7)$$

Concurrently for the matrix determinant of Equation (6), we have  $C_i$  and  $C'_i$ , which are the  $i$ -th column and its new  $i$ -th column after a column operation of  $|D|$ , respectively. By applying the row operation  $R'_{\frac{n}{2}+i} = R_{\frac{n}{2}+i} - R_i$ , following by  $C'_i = C_i + C_{\frac{n}{2}+i}$  on Equation (6), for every  $1 \leq i \leq \frac{n}{2}$ ,

then we see that

$$|D| = \begin{vmatrix} (\lambda - 1)I_{\frac{n}{2}} & -I_{\frac{n}{2}} \\ 0_{\frac{n}{2}} & (\lambda + 1)I_{\frac{n}{2}} \end{vmatrix}. \quad (8)$$

Due to Equation (8) being the determinant of an upper triangular matrix, then we have

$$|D| = \left| (\lambda - 1)I_{\frac{n}{2}} \right| \left| (\lambda + 1)I_{\frac{n}{2}} \right| = (\lambda + 1)^{\frac{n}{2}} (\lambda - 1)^{\frac{n}{2}}. \quad (9)$$

Using Equations (7) and (9), consequently

$$P_{\text{GA}(\Gamma_G)}(\lambda) = (\lambda + 1)^{\frac{3n-6}{2}} (\lambda - (n - 3)) (\lambda - 1)^{\frac{n}{2}}.$$

Here GA-spectrum of  $\Gamma_G$  is

$$\text{Spec}(\Gamma_G) = \left\{ (n - 3)^1, (1)^{\frac{n}{2}}, (-1)^{\frac{3n-6}{2}} \right\}.$$

In the end, by direct calculation we obtain

$$E_{\text{GA}}(\Gamma_G) = (1)|n - 3| + \frac{n}{2}|1| + \frac{3n - 6}{2}|-1| = 2(n - 3) + n. \quad (10)$$

■

### 3.2. Symmetric division deg energy

**Theorem 7.** Let  $G = G_1 \cup G_2$ . The symmetric division deg energy of  $\Gamma_G$  is

1. for  $n$  is odd,  $E_{\text{SDD}}(\Gamma_G) = 4(n - 2)$ , and
2. for  $n$  is even,  $E_{\text{SDD}}(\Gamma_G) = 4(n - 3) + 2n$ .

**Proof.**

1. By considering the properties from Theorem 2 (1) for the odd  $n$  that  $d_{a_i} = n - 2$  and  $d_{a_i b} = 0$ , for all  $i = 1, 2, \dots, n$  in conjunction with Definition 2 and Theorem 3 (1), similar to the labeling of  $G_1$  and  $G_2$  in Theorem 6 (1), then  $\text{SDD}(\Gamma_G)$  is a matrix with dimension  $(2n - 1) \times (2n - 1)$  as given below:

$$\text{SDD}(\Gamma_G) = \begin{bmatrix} 2(J - I)_{n-1} & 0_{(n-1) \times n} \\ 0_{n \times (n-1)} & 0_n \end{bmatrix}.$$

Then from Equation (1), one can easily see that,  $\text{SDD}(\Gamma_G) = 2 \cdot \text{GA}(\Gamma_G)$ . From the relation in Equation (2), the symmetric division deg energy of  $\Gamma_G$  is equivalent to

$$E_{\text{SDD}}(\Gamma_G) = 2 \cdot E_{\text{GA}}(\Gamma_G) = 2 \cdot 2(n - 2) = 4(n - 2).$$

2. Using the same reasoning as the proof of Theorem 6 (2) for  $n$  is even, we get the adjacency relation between two vertices by following Theorem 3 (2). Combining it with Theorem 2 (2) and Definition 2, clearly that we can construct  $\text{SDD}(\Gamma_G)$  of size  $(2n - 2) \times (2n - 2)$ :

$$\text{SDD}(\Gamma_G) = \begin{bmatrix} 2(J - I)_{n-2} & 0_{(n-2) \times \frac{n}{2}} & 0_{(n-2) \times \frac{n}{2}} \\ 0_{\frac{n}{2} \times (n-2)} & 0_{\frac{n}{2}} & 2I_{\frac{n}{2}} \\ 0_{\frac{n}{2} \times (n-2)} & 2I_{\frac{n}{2}} & 0_{\frac{n}{2}} \end{bmatrix} = 2 \cdot \text{GA}(\Gamma_G).$$

Consider the matrix in Equation (3), then  $\text{SDD}(\Gamma_G) = 2 \cdot \text{GA}(\Gamma_G)$ . Applying Equation (10), we obtain:

$$E_{\text{SDD}}(\Gamma_G) = 2 \cdot E_{\text{GA}}(\Gamma_G) = 2(2(n - 3) + n) = 4(n - 3) + 2n. \quad (11)$$

■

### 3.3. Degree exponent energy

**Theorem 8.** Let  $G = G_1 \cup G_2$ . The degree exponent energy for  $\Gamma_G$  is

1. for  $n$  is odd,  $E_{\text{DE}}(\Gamma_G) = 2(n - 2)^{n-1} + 2(n - 1)$ , and
2. for  $n$  is even,  $E_{\text{DE}}(\Gamma_G) = \begin{cases} 10, & \text{if } n = 4 \\ 2(n - 3)^{n-2} + 2(n - 1), & \text{if } n > 4. \end{cases}$

**Proof.**

1. From the first case  $n$  is odd, based on Theorems 2 (1), 3 (1) and Definition 3, we can construct  $\text{DE}(\Gamma_G)$  of order  $(2n-1) \times (2n-1)$ . We label  $G_1$  and  $G_2$  in the same way as proving Theorem 6(1). Therefore,

$$\text{DE}(\Gamma_G) = \begin{bmatrix} (n-2)^{n-2}(J-I)_{n-1} & J_{(n-1) \times n} \\ 0_{n \times (n-1)} & (J-I)_n \end{bmatrix}.$$

The characteristic formula of  $\text{DE}(\Gamma_G)$  can be states as

$$P_{\text{DE}(\Gamma_G)}(\lambda) = \begin{vmatrix} (\lambda + (n-2)^{n-2})I_{n-1} - (n-2)^{n-2}J_{n-1} & -J_{(n-1) \times n} \\ 0_{n \times (n-1)} & (\lambda+1)I_n - J_n \end{vmatrix}. \quad (12)$$

Let  $n_1 = n-1$ ,  $n_2 = n$  and  $a = (n-2)^{n-2}$ ,  $b = c = 1$ ,  $d = 0$ , then directly from Lemma 1, we can declare Equation (12) as

$$P_{\text{DE}(\Gamma_G)}(\lambda) = (\lambda + (n-2)^{n-2})^{n-2}(\lambda+1)^{n-1}(\lambda - (n-2)^{n-1})(\lambda - (n-1)).$$

Since DE-spectrum of  $\Gamma_G$  is

$$\text{Spec}(\Gamma_G) = \left\{ ((n-2)^{n-1})^1, (n-1)^1, (-1)^{n-1}, (-(n-2)^{n-2})^{n-2} \right\},$$

then we easily conclude that the degree exponent energy of  $\Gamma_G$  is

$$E_{\text{DE}}(\Gamma_G) = (n-2) |-(n-2)^{n-2}| + (n-1) |-1| + |(n-2)^{n-1}| + |n-1| = 2(n-2)^{n-1} + 2(n-1).$$

2. Similar discussion for the even  $n$ , again from Theorems 2 (2), 3 (2) and directly from Definition 3, then  $\text{DE}(\Gamma_G)$  is a  $(2n-2) \times (2n-2)$  matrix:

$$\text{DE}(\Gamma_G) = \begin{bmatrix} (n-3)^{n-3}(J-I)_{n-2} & (n-3)J_{(n-2) \times n} \\ J_{n \times (n-2)} & (J-I)_n \end{bmatrix}.$$

It follows that

$$P_{\text{DE}(\Gamma_G)}(\lambda) = \begin{vmatrix} (\lambda + (n-3)^{n-3})I_{n-2} - (n-3)^{n-3}J_{n-2} & -(n-3)J_{(n-2) \times n} \\ -J_{n \times (n-2)} & (\lambda+1)I_n - J_n \end{vmatrix}. \quad (13)$$

From Lemma 1, with  $a = (n-3)^{n-3}$ ,  $b = d = 1$ ,  $c = n-3$ ,  $n_1 = n-2$  and  $n_2 = n$ , then Equation (13) can be written as

$$P_{\text{DE}(\Gamma_G)}(\lambda) = (\lambda + (n-3)^{n-3})^{n-3}(\lambda+1)^{n-1} \\ \times (\lambda^2 - (n-1 + (n-3)^{n-2})\lambda + (n-1)(n-3)^{n-2} - n(n-2)(n-3)).$$

Note that the quadratic formula gives two conditions. For  $n = 4$ , in this situation, we have  $E_{\text{DE}}(\Gamma_G) = 10$ , whereas for  $n > 4$ , the DE-energy for  $\Gamma_G$  is

$$E_{\text{DE}}(\Gamma_G) = 2(n-3)^{n-2} + 2(n-1). \quad \blacksquare$$

### 3.4. Inverse sum indeg energy

**Theorem 9.** Let  $G = G_1 \cup G_2$ . The inverse sum indeg energy of  $\Gamma_G$  is

1. for  $n$  is odd,  $E_{\text{ISI}}(\Gamma_G) = (n-2)^2$ , and
2. for  $n$  is even,  $E_{\text{ISI}}(\Gamma_G) = (n-3)^2 + \frac{n}{2}$ .

**Proof.**

1. From the first case, when  $n$  is odd, in the same manner as proving Theorem 6 (1), and according to Definition 5, then we can construct  $\text{ISI}(\Gamma_G)$  of order  $(2n-1) \times (2n-1)$  as follows:

$$\text{ISI}(\Gamma_G) = \begin{bmatrix} \left(\frac{n-2}{2}\right)(J-I)_{n-1} & 0_{(n-1) \times n} \\ 0_{n \times (n-1)} & 0_n \end{bmatrix}. \quad (14)$$

In the light of the proofing part of Theorem 3.2 (1) in [14], then the immediate expression of Equation (14) is

$$\text{ISI}(\Gamma_G) = \frac{1}{2} \cdot \text{MaxD}(\Gamma_G) = \frac{1}{2} \cdot \text{MinD}(\Gamma_G),$$

which implies from Theorem 5, the inverse sum indeg energy of  $\Gamma_G$  is

$$E_{\text{ISI}}(\Gamma_G) = \frac{1}{2} \text{MaxD}(\Gamma_G) = \frac{1}{2} \cdot 2(n-2)^2 = (n-2)^2.$$

2. Again, in the same way as proving Theorem 6 (2) for  $n$  is even, and combining with Definition 4, clearly that we can construct a  $(2n-2) \times (2n-2)$  matrix,  $\text{ISI}(\Gamma_G)$  as follows:

$$\text{ISI}(\Gamma_G) = \begin{bmatrix} \frac{n-3}{2}(J-I)_{n-2} & 0_{(n-2) \times \frac{n}{2}} & 0_{(n-2) \times \frac{n}{2}} \\ 0_{\frac{n}{2} \times (n-2)} & 0_{\frac{n}{2}} & \frac{1}{2}I_{\frac{n}{2}} \\ 0_{\frac{n}{2} \times (n-2)} & \frac{1}{2}I_{\frac{n}{2}} & 0_{\frac{n}{2}} \end{bmatrix}. \quad (15)$$

Invoking the proofing part of Theorem 3.2 (2) in [14], in other words, Equation (15) can be written by

$$\text{ISI}(\Gamma_G) = \frac{1}{2} \cdot \text{MaxD}(\Gamma_G) = \frac{1}{2} \cdot \text{MinD}(\Gamma_G).$$

Therefore,

$$E_{\text{ISI}}(\Gamma_G) = \frac{1}{2} E_{\text{MaxD}}(\Gamma_G) = \frac{1}{2} (2(n-3)^2 + n) = (n-3)^2 + \frac{n}{2},$$

which, due to Theorem 5, proves the theorem. ■

### 3.5. Sombor energy

**Theorem 10.** Let  $G = G_1 \cup G_2$ . The Sombor energy of  $\Gamma_G$  is

1. for  $n$  is odd,  $E_S(\Gamma_G) = 2\sqrt{2}(n-2)^2$ , and
2. for  $n$  is even,  $E_S(\Gamma_G) = 2\sqrt{2}(n-3)^2 + \sqrt{2}n$ .

**Proof.**

1. Given  $n$  is odd, we have the same way of labeling  $G_1$  and  $G_2$  as a proof of Theorem 6 (1). By Definition 5, the properties of Theorem 2 (1), and also the form of  $\Gamma_G$  in Theorem 3 (1), then matrix  $S(\Gamma_G)$  of size  $(2n-1) \times (2n-1)$  is as given below:

$$S(\Gamma_G) = \begin{bmatrix} \sqrt{2}(n-2)(J-I)_{n-1} & 0_{(n-1) \times n} \\ 0_{n \times (n-1)} & 0_n \end{bmatrix}.$$

Combining the last matrix with the proofing part of Theorem 3.2 (1) in [14], we conclude that

$$S(\Gamma_G) = \sqrt{2} \cdot \text{MaxD}(\Gamma_G) = \sqrt{2} \cdot \text{MinD}(\Gamma_G).$$

Using Theorem 5 (1), from the above equation, we obtain the Sombor energy of  $\Gamma_G$ ,

$$E_S(\Gamma_G) = \sqrt{2} E_{\text{MaxD}}(\Gamma_G) = 2\sqrt{2}(n-2)^2.$$

2. For the second case, by the same method as a proof of Theorem 6 (2) and using Definition 5, we also obtain a similar result for even  $n$ . Then the Sombor matrix of  $\Gamma_G$  of size  $(2n-2) \times (2n-2)$  follows immediately,

$$S(\Gamma_G) = \begin{bmatrix} \sqrt{2}(n-3)(J-I)_{n-2} & 0_{(n-2) \times \frac{n}{2}} & 0_{(n-2) \times \frac{n}{2}} \\ 0_{\frac{n}{2} \times (n-2)} & 0_{\frac{n}{2}} & \sqrt{2}I_{\frac{n}{2}} \\ 0_{\frac{n}{2} \times (n-2)} & \sqrt{2}I_{\frac{n}{2}} & 0_{\frac{n}{2}} \end{bmatrix},$$

and  $S(\Gamma_G) = \sqrt{2} \cdot \text{MaxD}(\Gamma_G) = \sqrt{2} \cdot \text{MinD}(\Gamma_G)$  as a consequence of the proofing part of Theorem 3.2 (2) in [14]. Consider Theorem 5 (2), we then obtain

$$E_S(\Gamma_G) = \sqrt{2} E_{\text{MaxD}}(\Gamma_G) = 2\sqrt{2}(n-3)^2 + \sqrt{2}n. \quad \text{■}$$

## 4. Discussion

By inspecting the results in Theorems 6–10, it is possible to derive several conclusions from the given energies. The connection between energies yields the following two corollaries.

**Corollary 1.** *Let  $G = G_1 \cup G_2$ , then*

1.  $E_{\text{ISI}}(\Gamma_G) = \frac{\sqrt{2}}{2} \cdot E_S(\Gamma_G)$ .
2.  $E_{\text{SDD}}(\Gamma_G) = 2 \cdot E_{\text{GA}}(\Gamma_G)$ .
3.  $E_{\text{DE}}(\Gamma_G) = \frac{1}{2} \cdot E_{\text{DES}}(\Gamma_G)$ .

**Corollary 2.** *Let  $G = G_1 \cup G_2$ , then*

$$E_{\text{GA}}(\Gamma_G) < E_{\text{SDD}}(\Gamma_G) < E_{\text{ISI}}(\Gamma_G) < E_S(\Gamma_G) < E_{\text{DE}}(\Gamma_G) < E_{\text{DES}}(\Gamma_G).$$

For the commuting graph for dihedral groups, from Corollary 1, it can be seen that there are three clusters of the relationship between degree-based energies. Besides, the ordering of the obtained energies in Corollary 2 shows that the two largest ones are the degree exponent sum energy and degree exponent energy. The definition of DES and DE matrices consider the non-diagonal and diagonal entries, while the other matrix definitions require the adjacency relation between two vertices in  $\Gamma_G$ . We also conclude that the lowest is the geometric arithmetic energy.

Furthermore, the geometric-arithmetic energy, symmetric division deg energy, and degree exponent energies are always a positive even integer as well as the maximum degree energy, minimum degree energy, and degree exponent sum energy from the previous result. On the other hand, the inverse sum indeg energy is a positive integer that can be either even or odd. Meanwhile, the Sombor energy is never an odd integer. More specifically, this paper's energy formulas are never the square root of an odd integer, except ISI-energy.

## 5. Conclusion

In this work, we determined the explicit formulas of the degree-based energies of commuting graphs for dihedral groups and also obtained the comparison and the ordering of the given energies.

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## Енергії комутуючого графа для дієдричних груп на основі степенів

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Комутуючий граф для скінченної групи  $G$ , що позначається  $\Gamma_G$ , з множиною вершин  $G \setminus Z(G)$ , де  $Z(G)$  — центр  $G$ , є графом, в якому  $v_p, v_q \in G \setminus Z(G)$ ,  $v_p \neq v_q$  є суміжними, якщо  $v_p v_q = v_q v_p$ . Останніми роками проводилися значні дослідження енергії графів, зокрема з акцентом на матриці, пов'язані зі степенями вершин. Саме цим і було мотивоване наше дослідження, яке присвячене енергії  $\Gamma_G$  для дієдральних груп порядку  $2n$ ,  $D_{2n}$ , стосовно деяких графових матриць, пов'язаних зі степенем елементів  $D_{2n} \setminus Z(D_{2n})$ , та вивченню кореляції між цими енергіями. До розглянутих матриць належать геометрично-арифметична, симетрична матриця ділення степенів, матриця степені експонент, матриця оберненої суми степенів та матриця Сомбора. На основі цих п'яти матриць було встановлено, що найменшу енергію графа має геометрично-арифметична енергія  $\Gamma_G$ , тоді як найвищу — енергія показника степені. Крім того, геометрично-арифметична енергія, енергія симетричного ділення степенів та енергія показника степеня завжди є додатними парними цілими числами. Натомість, енергія оберненої суми степенів є додатним цілим числом, яке може бути як парним, так і непарним. Енергія Сомбора ніколи не є непарним цілим числом.

**Ключові слова:** *комутуючий граф; енергія графа; матриці на основі степенів; дієдральна група.*